The K-theory of cohomogeneity-one actions (clean)

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Abstract

We compute the equivariant complex K-theory ring of a cohomogeneity-one action of a compact Lie group at the level of generators and relations and derive a characterization of K-theoretic equivariant formality for these actions. Less explicit expressions survive for a range of equivariant cohomology theories including Bredon cohomology and Borel complex cobordism. The proof accordingly involves elements of equivariant homotopy theory, representation theory, and Lie theory.

Aside from analysis of maps of representation rings and heavy use of the structure theory of compact Lie groups, a more curious feature is the essential need for a basic structural fact about the Mayer–Vietoris sequence for any multiplicative cohomology theory which seems to be otherwise unremarked in the literature, and a similarly unrecognized basic lemma governing the equivariant cohomology of the orbit space of a finite group action.

Compact Lie group actions $G \curvearrowright M$ of *cohomogeneity one*, those whose orbit space M/G is a 1-manifold, have been a perennial object of study in differential geometry [Mos57a, Neu68, Par86, AlAl93, Püt09, Hoel10, Fra11, He14, GaZ18, AnP20], first because they are the most obvious class to study after homogeneous (= cohomogeneity-zero) actions, but also because they furnish examples of Einstein metrics [Ber82] and manifolds with exceptional holonomy [BryS89, CGLP02, CGLP04], and especially because "large" isometry, for which low cohomogeneity gives a measure, has long played a central organizing role (sometimes called the *Grove program* [Grove]) in finding Riemannian manifolds of nonnegative curvature [GrZ00, GrZ02, Ver04, GrVWZ06, GrWZ08, Zil09, Dear11, VZ14]. As nontrivial amounts of work have gone into understanding these actions geometrically,[1] their algebro-topological invariants are of some interest, and phenomena arising in the computation of the rational Borel equivariant cohomology of these actions [CGHM19] hint at the generalization to a large class of cohomology theories pursued in the present work. The case of equivariant K-theory is particularly interesting, given its implications for the existence of vector bundles with prescribed properties; for example, Theorem 6.1 of the present work is used in a work of Amann–González-Álvaro–Zibrowius [AmGÁZ19, Thm. A(1)] to construct metrics of non-negative curvature on vector bundles over a class of manifolds admitting cohomogeneity-one actions.

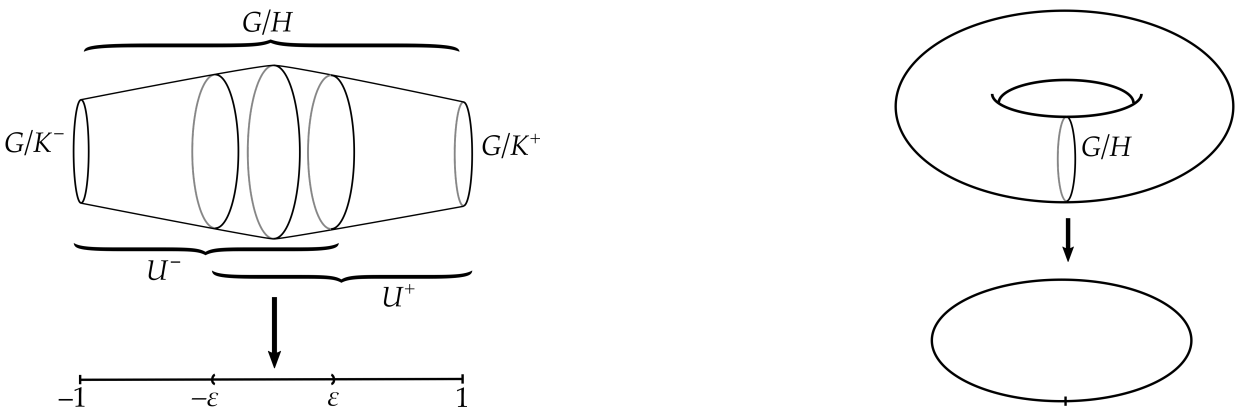
In considering cohomogeneity-one actions, one almost always operates in the framework of Mostert's classical structure theorem[2], encapsulated in Figure 0.2.

[1] See the bibliography in the recent work of Galaz-García and Zarei [GaZ18] for some indication of the scope of this study.

[2] with an important erratum caught by Richardson and Samelson [Mos57b]

Theorem 0.1 (Mostert [Mos57a]). *Let G be a compact Lie group acting smoothly on a compact smooth manifold M in such a way that the quotient M/G is a compact, connected 1-manifold, possibly with boundary.*[3]

- *If M/G is a closed interval, there are inclusions of closed subgroups $H \rightrightarrows K^{\pm} \rightrightarrows G$ such that $K^{\pm}/H$ are homeomorphic to spheres*[4] *and M is the double mapping cylinder of the span $G/H \rightrightarrows G/K^{+}$.*
- *If M/G is a circle, there exist a closed subgroup H of G and an element w of the normalizer $N_G(H)$ such that M is diffeomorphic to the mapping torus of the right translation by w on G/H.*



(a) M/G an interval **(b)** M/G a circle

Figure 0.2: Schematics for the orbit projection $M \longrightarrow M/G$ of a cohomogeneity-one action

In the case of the double mapping cylinder, if M is smooth, then the isotropy quotients $K^{\pm}/H$ can actually be taken isometric in the Riemannian sense to round spheres given by orbits in irreducible $K^{\pm}$-representations [Besse, Ex. 7.13], suggesting equivariant complex K-theory K_G^*, whose coefficient ring is the ring RG of complex representations, which is already motivated by its applications, is also an especially natural topological invariant of such an action. Indeed, the Mayer–Vietoris sequence of the cover $\{U^{\pm}\}$ in Figure 0.2(a) reduces to the exact sequence

$$0 \to K_G^0(M) \longrightarrow RK^{-} \times RK^{+} \longrightarrow RH \xrightarrow{\delta} K_G^1(M) \to 0, \qquad (0.3)$$

where the middle map is the difference of the restrictions $RK^{\pm} \longrightarrow RH$ between complex representation rings, showing the additive structure of $K_G^*(M)$ is wholly a question of representation theory.

Surprisingly, the multiplicative structure turns out to be as well. The key fact is that the connecting map δ in (0.3) is actually a $K_G^0(M)$-module homomorphism. The analogous fact in Borel cohomology can be established by chasing cochains around a diagram, but there are no cochains to follow in K-theory. The result nevertheless turns out to be extremely general:

Proposition 2.1. *Let E^* be a multiplicative ($\mathbb{Z}$-graded, G-equivariant) cohomology theory. Then the natural $E^*(X)$-module structure on the terms of the Mayer–Vietoris sequence of a triad $(X; U, V)$ of G–CW complexes with $X = U \cup V$ is preserved by the connecting map in the sequence.*

[3] In the noncompact case, where the quotient space is an open or half-open interval, M deformation retracts onto a homogeneous fiber G/H of $M \longrightarrow M/G$, so this case is already understood from the point of view of this paper.

[4] Without the smoothness hypothesis (omitted by Mostert), $K^{\pm}/H$ can also be the Poincaré homology sphere, as noted by Galaz-García and Zarei only recently [GaZ18].

This basic result seems underappreciated; working topologists surveyed by the author seem not to know it, nor does it seem to be discussed in the literature. The enhanced connecting map makes life simpler in a variety of situations, and a sample application to the cup product on a closed 3-manifold is discussed in Example 2.2. Most importantly for us, Proposition 2.1 immediately implies a general structure theorem for the equivariant cohomology ring of $G \curvearrowright M$ in multiplicative cohomology theories with coefficients concentrated in even degree, Proposition 2.9, and one thus has a general expression for the K-theory ring, Theorem 2.11.

To say more concretely what the ring $K^*_G(M)$ is, one needs to explicitly identify the maps in the sequence (0.3). The structure theorem for $H^*_G(M;\mathbb{Q})$ proceeds from analysis of an analogous sequence, so one naturally changes the nouns in those statements and hopes the same arguments will prove the stronger results. While the results are indeed the expected ones, the cohomological proof methods fail utterly and the K-theoretic proof is incomparably more involved.

For example, the algebraic lemma governing the map $H^*(BK;\mathbb{Q}) \longrightarrow H^*(BH;\mathbb{Q})$ when K/H is an odd-dimensional sphere is an easy result on commutative graded algebras, but the analogous statement about surjections $RK \longrightarrow RH$ between ungraded polynomial rings is a deep open problem in affine algebraic geometry, the *Abhyankar–Sathaye embedding conjecture*, and one is forced to an analysis in Section 4 involving the structure theory of compact Lie groups and the classification of homogeneous spheres. The result when one of the spheres $K^\pm/H$ is odd-dimensional then follows:

Theorem 4.1. *Let M be the double mapping cylinder of the span $G/H \rightrightarrows G/K^\pm$ for inclusions $H \rightrightarrows K^\pm \rightrightarrows G$ of closed, connected subgroups of a compact Lie group G such that $K^\pm/H$ are spheres and the fundamental groups $\pi_1(K^\pm)$ are free abelian.*

(a) Assume that both K^+/H and K^-/H are odd-dimensional. Then we have an RG-algebra isomorphism of $K^*_G(M) = K^0_G(M)$ with one of

$$\frac{RH[t_-^{\pm 1}, t_+^{\pm 1}]}{(t_- - 1)(t_+ - 1)}, \qquad \frac{RH[t_-^{\pm 1}, \overline{\rho}_+]}{(t_- - 1)(\overline{\rho}_+)}, \qquad \frac{RH[\overline{\rho}_-, t_+^{\pm 1}]}{(\overline{\rho}_-)(t_+ - 1)}, \qquad \frac{RH[\overline{\rho}_-, \overline{\rho}_+]}{(\overline{\rho}_- \overline{\rho}_+)},$$

where we identify $RK^\pm$ with the Laurent polynomial ring $RH[t_\pm^{\pm 1}]$ when $\dim K^\pm/H = 1$ and with the polynomial ring $RH[\overline{\rho}_\pm]$ when $\dim K^\pm/H \geqslant 3$.

(b) Assume K^+/H is odd-dimensional and K^-/H is even-dimensional. Then we have an RG-algebra isomorphism of $K^*_G(M) = K^0_G(M)$ with

$$RK^- \oplus (t-1)RH[t^{\pm 1}] < RH[t^{\pm 1}] \cong RK^+ \qquad \text{or} \qquad RK^- \oplus \overline{\rho} RH[\overline{\rho}] < RH[\overline{\rho}] \cong RK^+,$$

where we identify RK^+ with $RH[t_\pm^{\pm 1}]$ if $\dim K^+/H = 1$ and with $RH[\overline{\rho}_\pm]$ if $\dim K^+/H \geqslant 3$. The product in either case is determined by the restriction $RK^- \rightarrowtail RH$.

In all cases the RG-module structure is determined by restriction.

Similar difficulties ensue when the spheres $K^\pm/H$ are both even-dimensional. The determination of the product on $H^*_G(M;\mathbb{Q})$ in this case reduces to pleasant arguments involving Serre spectral sequences of fibrations between classifying spaces and the eigenspaces of the action of the so-called *Weyl group of a geodesic* of M on $H^*(BH;\mathbb{C})$, relying on the fact these eigenspaces are themselves graded vector spaces; but the proof in K-theory involves a lengthy multi-layered induction on the structure of compact Lie groups, whose base cases require a number of lemmas in the Lie theory and representation theory of simple Lie groups. The result, however, comes out as clean as one could hope:

Theorem 0.4. *Let M be the double mapping cylinder of the span $G/H \rightrightarrows G/K^{\pm}$ for inclusions $H \rightrightarrows K^{\pm} \rightrightarrows G$ of compact Lie groups such that $K^{\pm}$ are semisimple groups which are products of simply-connected groups and* $\mathrm{SO}(\mathrm{odd})$ *factors and $K^{\pm}/H$ are even-dimensional spheres. Then there exist an element $z \in K^1_G(M)$ and an RG-algebra isomorphism*

$$K^*_G(M) \cong (RK^-|_H \cap RK^+|_H) \otimes \Lambda[z],$$

where the injections $RK^{\pm} \longrightarrow RH$ and the RG-module structure are given by restriction.

This statement is a simplification of the more general but less pithy Theorem 5.10. The base cases of the induction remarkably all turn out to be known special examples; see Remark 5.11.

These structure results also allow one to characterize surjectivity of the map $K^*_G(M) \longrightarrow K^*(M)$, also known as K-theoretic *equivariant formality*, using the Hodgkin–Künneth and Atiyah–Hirzebruch–Leray–Serre spectral sequences and some homological algebra:

Theorem 6.1. *Consider a cohomogeneity-one action of a compact, connected Lie group G with $\pi_1(G)$ torsion-free on a smooth closed manifold M such that the orbit space M/G is an interval and the commutator subgroups of the exceptional isotropy groups $K^{\pm}$ are the products of simply-connected groups and* $\mathrm{SO}(\mathrm{odd})$ *factors. Then the action is K-theoretically equivariantly formal if and only if* $\operatorname{rk} G = \max\{\operatorname{rk} K^-, \operatorname{rk} K^+\}$.

So much for the case when M/G is an interval. When M/G is a circle, we can say nothing categorical before inverting the order $|\Gamma|$ of the cyclic subgroup Γ generated by the class of $w \in N_G(H)$ in the component group $\pi_0 N_G(H)$ (see Example 1.9), but once we do, the result follows formally from a much more fundamental fact about equivariant cohomology theories:

Theorem 1.2. *Let G be a compact Lie group and Γ a discrete finite group, and X a finite $(G \times \Gamma)$–CW complex whose isotropy subgroups are of the form $H \times \Delta$ for $H \leqslant G$ and $\Delta \leqslant \Gamma$. Moreover, let E^* be a $\mathbb{Z}$-graded G-equivariant cohomology theory valued in $\mathbb{Z}[1/|\Gamma|]$-modules. Then the quotient map $\pi\colon X \longrightarrow X/\Gamma$ induces an isomorphism*

$$E^*(X/\Gamma) \xrightarrow{\sim} E^*(X)^{\Gamma}$$

onto the submodule of Γ-invariant elements.

The proof uses an equivariant Atiyah–Hirzebruch spectral sequence and an observation about Bredon cohomology to reduce to the classical result for singular cohomology it generalizes, and the result is again the sort of thing that one expects to find in the literature but does not. In any event, it has an immediate corollary, Lemma 1.5, describing the equivariant cohomology of a mapping torus in broad generality, which specializes to the result we wanted:

Proposition 1.7. *Let M be the mapping torus of the right translation by $w \in N_G(H)$ on a homogeneous space G/H of a Lie group G with finitely many components, and write w^* for the maps induced on $K^*(G/H)$ and $K^*_G(G/H) \cong RH$ by the right translation by w. Let ℓ be the least positive natural number such that w^ℓ lies in the identity component of $N_G(H)$. Then one has $K^*(S^1)$- and $\big(RG \otimes K^*(S^1)\big)$-algebra isomorphisms*

$$K^*_G\big(M; \mathbb{Z}[1/\ell]\big) \cong K^*(S^1) \otimes (RH)^{\langle r^*_w \rangle} \otimes \mathbb{Z}[1/\ell],$$

$$K^*\big(M; \mathbb{Z}[1/\ell]\big) \cong K^*(S^1) \otimes K^*(G/H)^{\langle r^*_w \rangle} \otimes \mathbb{Z}[1/\ell],$$

respectively, where $(-)^{\langle w^* \rangle}$ *denotes the subring of* w^**-invariant elements, the* $K^*(S^1)$*-module structure is given in both cases by pullback from* $M/G \approx S^1$*, and the* RG*-algebra structure is induced by the inclusion* $H \hookrightarrow G$*.*

The structure of the paper is as follows. The less involved case where M/G is a circle, including Proposition 1.7, along with some necessary definitions, is discussed in Section 1. In Section 2, we assume the orbit space M/G is an interval and discuss those aspects of $K^*_G(M)$ which do not depend on representation theory on the dimensions of the homogeneous spheres $K^\pm/H$, including the Mayer–Vietoris proposition 2.1 and a general structure theorem 2.11. The refinements of this theorem in the case M/G is an interval, depending on the parities of the dimensions of $K^\pm/H$, rely on material on Weyl groups, Lie theory, and maps of representation rings developed in Section 3. In Section 4, we derive the consequences, including Theorem 4.1, when one of the spheres $K^\pm/H$ is odd-dimensional, and in Section 5, we address the case when both of the spheres $K^\pm/H$ are even-dimensional and derive Theorem 0.4. Finally, in Section 6 we use these structural results to characterize K-theoretic equivariant formality for actions with orbit space an interval.

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1. Coverings and mapping tori

We begin with this section because it is the only one involving any inversion of coefficients or any specifically equivariant homotopy theory. It does not involve representation theory or Lie theory in any serious way, so it is somewhat independent of the rest of the document, and we take it as an opportunity to get some long definitions out of the way.

Recall from Theorem 0.1 that if a compact Lie group acts smoothly on a compact manifold M with orbit space a circle (the case in Figure 0.2(b)), then M is diffeomorphic to the ***mapping torus*** of the right translation by some element $w \in N_G(H)$ on G/H, namely

$$\frac{G/H \times [0,1]}{(gH,1) \sim (gwH,0)}.$$

As w is of finite order $|w|$, cutting the mapping torus at $t = 1$, gluing $|w|$ copies end to end, and then regluing the fiber $t = 0$ to $t = |w|$ by $w^{|w|} = \mathrm{id}_{G/H}$, we see $G/H \times S^1$ is a $|w|$-sheeted covering of M. The G-equivariant K-theory of $G/H \times S^1$ is easy to compute, so most of our work is in computing the equivariant cohomology of a space from that of a finite-sheeted cover.

Definition 1.1. Let G be a topological group. A ***G-n-cell*** is a space $G/K \times D^n$, where $K \leqslant G$ is a closed subgroup and D^n the closed n-disc, equipped with the G-action $g \cdot (hK, x) := (ghK, x)$. A ***$G$–CW complex*** is a G-space X constructed iteratively as the colimit (= union) of a sequence of spaces X_n, where X_0 is a disjoint union of G-0-cells and otherwise each X_n is obtained from X_{n-1} by adjoining a collection of G-n-cells $G/K_\alpha \times D^n_\alpha$ along G-equivariant attaching maps $G/K_\alpha \times S^{n-1}_\alpha \longrightarrow X_{n-1}$. When we do not specify otherwise, S^n comes equipped with the *trivial* G-action (and hence is, if you like, a G–CW complex each of whose G-cells is of the form $G/G \times D^k$). A ***G–CW pair*** (X, A) comprises a G–CW complex X and a G–CW subcomplex A, meaning each

G-cell of A is also a G-cell of X. Given a G–space X, we denote by $X_+ := X \amalg *$ the disjoint union of X and a new isolated, G-fixed point $*$.

A ***reduced G-equivariant*** ($\mathbb{Z}$-graded) ***cohomology theory*** is a contravariant graded abelian group–valued homotopy functor $\widetilde{E}^* = \bigoplus_{n\in\mathbb{Z}} \widetilde{E}^n$ on the category of pointed G–CW complexes which takes a cofiber sequence $A \to X \to X/A$ to an exact sequence of groups and is equipped with a natural graded group isomorphism $\sigma\colon \widetilde{E}^* X \xrightarrow{\sim} \widetilde{E}^{*+1}\Sigma X$ of degree one, the ***suspension***, where $\Sigma X = S^1 \wedge X$ is the reduced suspension of X. (Possibly obscured in the notation: S^1 is again assumed to have trivial G-action.) Such a theory comes automatically with an associated *unreduced theory* on unpointed G–CW pairs given by $E^*(X, A) := \widetilde{E}^*(X/A)$ (by convention $X/\varnothing := X_+$) and satisfying the Eilenberg–Steenrod axioms save dimension [Matu73, §1].

Let Orb_G denote the category of orbits G/K (for K closed) and G-equivariant maps, $h\mathsf{Orb}_G$ the category with the same objects but morphisms G-homotopy classes of G-maps, Top the category of topological spaces, and Ab the category of abelian groups. A ***coefficient system*** is a contravariant functor $M\colon h\mathsf{Orb}_G \longrightarrow \mathsf{Ab}$. For a given space X, the fixed point set assignment $G/H \longmapsto X^H$ gives a standard contravariant functor $\mathsf{Orb}_G \longrightarrow \mathsf{Top}$ and composing any covariant functor $\mathsf{Top} \longrightarrow \mathsf{Ab}$ gives a coefficient system. As an example, for each $n \in \mathbb{N}$ and each G–CW complex X there is a functor $\underline{H}_n(X)\colon G/H \longmapsto H_n(X^H_n, X^H_{n-1})$. The assignment $X \longmapsto \underline{H}_n(X)$ is itself covariantly functorial in G–CW complexes.

The ***Bredon cohomology*** $H^*_G(X; M)$ of a G–CW complex X with coefficients in a coefficient system M is defined as the cohomology of the complex $C^n_G(X; M) := \mathrm{Nat}\big(\underline{H}_n(X), M\big)$ of natural transformations $\underline{H}_n \longrightarrow M$, where the n^{th} coboundary map of the complex is precomposition with the tuple $\partial_n = (\partial_n^{G/H})_{G/H\in\mathsf{Orb}_G}$ for $\partial_n^{G/H}$ the connecting map in the long exact homology sequence of the triple $(X^H_{n+1}, X^H_n, X^H_{n-1})$. Bredon cohomology is the unique unreduced G-equivariant cohomology theory E^* which satisfies the wedge axiom and the requirement that $E^*(G/H) = E^0(G/H) = M(G/H)$ for $G/H \in \mathsf{Orb}_G$.

We write $|\Gamma|$ for the order of a group Γ.

Theorem 1.2. *Let G be a compact Lie group and Γ a discrete finite group, and X a finite $(G\times\Gamma)$–CW complex whose isotropy subgroups are of the form $H \times \Delta$ for $H \leqslant G$ and $\Delta \leqslant \Gamma$. Moreover, let E^* be a $\mathbb{Z}$-graded G-equivariant cohomology theory valued in $\mathbb{Z}[1/|\Gamma|]$-modules. Then the quotient map $\pi\colon X \longrightarrow X/\Gamma$ induces an isomorphism*

$$E^*(X/\Gamma) \xrightarrow{\sim} E^*(X)^\Gamma$$

onto the submodule of Γ-invariant elements.

Proof. We first show the result for Bredon cohomology $H^p(-; E^q)$. As the group $E^q(G/K)$ admits division by $|\Gamma|$, a classical Leray spectral sequence argument (apparently due to Grothendieck [Grot57, Thm. 5.3.1, Cor. to Prop. 5.2.3]) shows

$$\phi_{G/K}\colon H^*\big(X^K_p/\Gamma, X^K_{p-1}/\Gamma; E^q(G/K)\big) \longrightarrow H^*\big(X^K_p, X^K_{p-1}; E^q(G/K)\big)^\Gamma$$

is an isomorphism. Endow $E^q(G/K)$ with the trivial Γ-action. Since the Kronecker pairing is Γ-invariant, the universal coefficient morphism

$$H^p\big(X^K_p, X^K_{p-1}; E^q(G/K)\big) \twoheadrightarrow \mathrm{Hom}\,\big(H_p(X^K_p, X^K_{p-1}), E^q(G/K)\big)$$

is also Γ-equivariant, and since $E^q(G/K)$ is divisible by $|\Gamma|$, induces a surjection of Γ-invariants, every Γ-invariant element being the average over a Γ-orbit. It follows from this surjectivity, the surjectivity of $\phi_{G/K}$, and the functoriality of the universal coefficient theorem that

$$f_{G/K}\colon \operatorname{Hom}\big(H_p(X_p^K/\Gamma, X_{p-1}^K/\Gamma), E^q(G/K)\big) \longrightarrow \operatorname{Hom}\big(H_p(X_p^K, X_{p-1}^K), E^q(G/K)\big)^\Gamma$$

is also a surjection. By the observation that $\big(\frac{G\times\Gamma}{H\times\Delta}\big)^K/\Gamma \approx \Big(\big(\frac{G\times\Gamma}{H\times\Delta}\big)/\Gamma\Big)^K$, our assumption on the isotropy groups of X, and induction, we have $(X_n)^K/\Gamma = (X_n/\Gamma)^K$ for all n, so the natural transformations $\underline{H}_p(X/\Gamma) \longrightarrow E^q$ are encoded by coherent sequences in the domain of $\prod_{G/K\in\mathsf{Orb}_G} f_{G/K}$. Equally, assigning each $E^q(G/K)$ the trivial Γ-action, the Γ-equivariant natural transformations $\underline{H}_p(X) \longrightarrow E^q$ are coherent sequences in the codomain of $\prod_{G/K\in\mathsf{Orb}_G} f_{G/K}$. Thus we will have an isomorphism $C^p_G(X/\Gamma; E^q) \xrightarrow{\sim} C^p_G(X; E^q)^\Gamma$ if we can show $f_{G/K}$ is also injective for each $G/K \in \mathsf{Orb}_G$.

To this end we may forget the corestriction to Γ-invariants in the codomain and just show the map of Homs is injective, and for this it is enough to see the predual

$$\psi_{G/K}\colon H_p(X_p^K, X_{p-1}^K) \longrightarrow H_p\big((X_p/\Gamma)^K, (X_{p-1}/\Gamma)^K\big)$$

is surjective. From the definition of a $(G\times\Gamma)$–CW complex and our assumption on isotropy groups, the quotient $X_p^K/X_{p-1}^K = (X_p/X_{p-1})^K$ is a wedge of summands

$$(G/H_\alpha \times \Gamma/\Delta_\alpha)_+^K \wedge S^p = \big((G/H_\alpha)^K \times \Gamma/\Delta_\alpha\big)_+ \wedge S^p$$

for various product subgroups $H_\alpha \times \Delta_\alpha \leqslant G\times\Gamma$, so the group $H_p(X_p^K, X_{p-1}^K) \cong \widetilde{H}_p(X_p^K/X_{p-1}^K)$ decomposes as

$$\bigoplus_\alpha \widetilde{H}_p\Big(\big((G/H_\alpha)^K \times \Gamma/\Delta_\alpha\big)_+ \wedge S^p\Big) \cong \bigoplus_\alpha \widetilde{H}_0\big((G/H_\alpha)^K \times \Gamma/\Delta_\alpha\big)_+ \cong \bigoplus_\alpha H_0^*\big((G/H_\alpha)^K\big)^{\oplus|\Gamma/\Delta_\alpha|},$$

and quotienting by Γ we have a similar isomorphism

$$H_p\big((X_p/\Gamma)^K, (X_{p-1}/\Gamma)^K\big) \cong \bigoplus_\alpha H_0\big((G/H_\alpha)^K\big).$$

But under these identifications the α^{th} summand of $\psi_{G/K}$ is just iterated addition $(x_1,\ldots,x_{|\Gamma/\Delta_\alpha|}) \longmapsto x_1+\cdots+x_{|\Gamma/\Delta_\alpha|}$ in the group $H_0\big((G/H_\alpha)^K\big)$, which is certainly surjective.

Varying p, we have our isomorphism of cochain complexes $C^*_G(X/\Gamma; E^*) \longrightarrow C^*_G(X; E^*)^\Gamma$. Note that $C^*_G(X; E^*)$ is divisible by $|\Gamma|$ and recall that given a cochain complex C of $|\Gamma|$-divisible Γ-modules, the inclusion $C^\Gamma \hookrightarrow C$ induces an isomorphism $H^*(C^\Gamma) \xrightarrow{\sim} H^*(C)^\Gamma$ and multiplication by $|\Gamma|$ is again invertible on $H^*(C)$. Finally the composite

$$H^*_G(X/\Gamma; E^*) \xrightarrow{\sim} H^*\big(C^*(X;E^*)^\Gamma\big) \xrightarrow{\sim} H^*(X;E^*)^\Gamma$$

is the claimed isomorphism in Bredon cohomology.

There is a equivariant Atiyah–Hirzebruch spectral sequence due to Matumoto [Matu73, §4],[5] functorial in and converging to the E^*-cohomology of finite G–CW complexes, and the entries $E_2^{p,q}$ of its second page are the Bredon cohomology groups $H^p_G(-;E^q)$ with coefficients in the

[5] The spectral sequence with sheaf coefficients due to Segal [Seg68, §5] reduces to this one in the case $E^* = K^*_G$ but is less immediately adapted to our needs.

coefficient system $K \longmapsto E^q(G/K)$. Forgetting the Γ-action and regarding X as a G–CW complex, we see $\pi\colon X \longrightarrow X/\Gamma$ induces a morphism of these spectral sequences. Since the spectral sequence can be defined using a Cartan–Eilenberg $H(p,q)$-system with $H(p,q) := \bigoplus_n E^n(X_{p-1}, X_{q-1})$ and the skeleta X_j are Γ-invariant by definition, the differentials d_r of this spectral sequence are Γ-equivariant. On E_2 pages, the induced map of spectral sequences is $H^*_G(X/G; E^*) \longrightarrow H^*_G(X; E^*)$, which we have just seen is an isomorphism onto its image $H^*_G(X; E^*)^\Gamma$. Inductively applying the recollection about invariants of cochain complexes from the previous paragraph to each page, we see π^* induces a pagewise isomorphism of one spectral sequence with the Γ-invariants of the second, and so at E_∞ we recover an isomorphism $\operatorname{gr} E^*(X/\Gamma) \xrightarrow{\sim} (\operatorname{gr} E^*X)^\Gamma$, where gr denotes the associated graded module with respect to the cellular filtration. But for any filtered Γ-module N divisible by $|\Gamma|$, the inclusion $N^\Gamma \longhookrightarrow N$ induces an isomorphism $\operatorname{gr}(N^\Gamma) \xrightarrow{\sim} (\operatorname{gr} N)^\Gamma$, so the E_∞ map further factors through an isomorphism $\operatorname{gr} E^*(X/\Gamma) \xrightarrow{\sim} \operatorname{gr}\big(E^*(X)^\Gamma\big)$. This is the associated graded map induced by $E^*(X/\Gamma) \longrightarrow E^*(X)^\Gamma$, so as the filtration involved is finite, that map is an isomorphism as well [Board99, Thm. 2.6]. □

As a corollary we have a result on mapping tori, which we prefer to state as a ring isomorphism, so we will need to define an additional notion.

Definition 1.3. A G-equivariant cohomology theory E^* is said to be ***multiplicative*** if E^* is valued in commutative graded algebras and the suspension axiom is replaced in the following way. Note that $E^*(*, \varnothing) = \widetilde{E}^0 S^0$ is a commutative ring with unity 1 and the projections $\pi_Y, \pi_X\colon Y \times X \longrightarrow Y, X$ induce a natural *cross product*

$$\begin{aligned} \widetilde{E}^*Y \otimes \widetilde{E}^*X &\xrightarrow{\times} \widetilde{E}^*(Y \wedge X), \\ y \otimes x &\longmapsto \pi_Y^* y \cdot \pi_X^* x. \end{aligned}$$

The new axiom is that there exist an element $\varsigma \in \widetilde{E}^1 S^1$ such that the map $\sigma\colon \widetilde{E}^*X \xrightarrow{\sim} \widetilde{E}^{*+1}(S^1 \wedge X)$ given by $\sigma(x) := \varsigma \times x$ is a natural isomorphism.

Remark 1.4. This is somewhat leaner than the usual axiomatization. It is typical in defining a multiplicative cohomology theory to demand it be represented by a ring spectum, but we do not require our theories to satisfy the wedge axiom, and thus our results will allow for things like p-completed theories.

For non-represented theories, it is usual to require natural cross products satisfying naturality axioms, but it seems simpler to demand cup products and instead note the other axioms follow from the CGA structure and functoriality. The typical axiomization also demands sign-commutativity of evident squares involving suspensions, but these are all consequences of graded commutativity and the uniform definition of suspension as a cross product. Unreduced theories additionally require the cross product cooperate with the connecting maps from the long exact sequences of a pair, but the connecting map can be defined in terms of the suspension in the unreduced theory, so the commutativity of these squares is again a formal consequence of functoriality and the uniform definition of the suspension.

Now we can state the result.

Lemma 1.5. *Let Y be a G-space and φ a self-homeomorphism of Y commuting with the G-action and such that there exists a positive integer ℓ such that φ^ℓ is homotopic to id_Y. Write X for the mapping torus of*

φ and let E^* be a $\mathbb{Z}$-graded multiplicative equivariant cohomology theory valued in $\mathbb{Z}[1/\ell]$-algebras. Write $E^* := E^*(*)$. Then

$$E^*X \cong E^*(Y)^{\langle \varphi^* \rangle} \underset{E^*}{\otimes} \Lambda_{E^*}[z],$$

where z is the pullback of a generator of $\widetilde{E}^1(S^1) \cong \widetilde{E}^0(S^0) = E^*$ under $X \longrightarrow S^1$.

Here, as usual, $E^*(Y)^{\langle \varphi^* \rangle}$ denotes the subring of elements invariant under pullback by φ.

Proof. Note that X admits an ℓ-sheeted cyclic covering Z by the mapping torus of φ^ℓ, which is homotopy equivalent to the mapping torus $Y \times S^1$ of the identity. The homotopy equivalence $h\colon Z \longrightarrow Y \times S^1$ and its homotopy inverse j can both be taken to preserve the projection to S^1, and the action of $g = 1 + \ell\mathbb{Z} \in \mathbb{Z}/\ell$ on Z induces a map hgj on $Y \times S^1$ which is homotopic to $(y, \theta) \longmapsto \big(\varphi(y), \theta + \frac{2\pi}{\ell}\big)$, which, rotating the S^1 component, is in turn homotopic to $(y, \theta) \longmapsto \big(\varphi(y), \theta\big)$. It follows from the suspension axiom for $\widetilde{E}^*$ that $E^*S^1 \cong E^* \oplus \widetilde{E}^*S^1 \cong E^* \oplus E^*[1] \cong E^* \oplus E^* \cdot \{z\}$. Now assuming multiplicativity, as $z \in E^1S^1$ is a free $E^*(*)$-module generator of $\widetilde{E}^*S^1$ and S^1 is a suspension, we have $E^*S^1 \cong \Lambda_{E^*}[z]$. It follows again from the suspension axiom that $E^*S^1 \otimes_{E^*} E^*Y \longrightarrow E^*(S^1 \times Y)$ is a ring isomorphism.[6] The action of $1 + \ell\mathbb{Z}$ on $E^*Y \otimes_{E^*} E^*S^1 \cong E^*Z$ is given by $a \otimes s \longmapsto \varphi^*a \otimes s$, so an application of Theorem 1.2 yields the claim. □

Proposition 1.6. *Let a cohomogeneity-one action of a compact, connected Lie group G on a smooth manifold M be given with orbit space $M/G \approx S^1$. Recall from Theorem 0.1 that this means M is G-equivariantly diffeomorphic to the mapping torus of right multiplication on G/H by some element $w \in N_G(H)$ and let ℓ be the smallest positive integer such that w^ℓ lies in the identity component of $N_G(K)$. Suppose E^* is a $\mathbb{Z}$-graded multiplicative equivariant cohomology theory valued in $\mathbb{Z}[1/\ell]$-algebras. Then one has a graded ring isomorphism*

$$E^*M \;\cong\; E^*(G/H)^{\langle r_w^* \rangle} \underset{E^*}{\otimes} \Lambda_{E^*}[z_1], \qquad |z_1| = 1.$$

Proof. Note that w^ℓ lies in the path-component of the identity, so that right multiplication by w^ℓ is homotopic to $\mathrm{id}_{G/H}$, and apply Lemma 1.5. □

The result we want follows immediately:

Proposition 1.7. *Let M be the mapping torus of the right translation by $w \in N_G(H)$ on a homogeneous space G/H of a Lie group G with finitely many components, and write w^* for the maps induced on $K^*(G/H)$ and $K^*_G(G/H) \cong RH$ by the right translation by w. Let ℓ be the least positive natural number such that w^ℓ lies in the identity component of $N_G(H)$. Then one has $K^*(S^1)$- and $\big(RG \otimes K^*(S^1)\big)$-algebra isomorphisms*

$$K^*_G\big(M; \mathbb{Z}[1/\ell]\big) \cong K^*(S^1) \otimes (RH)^{\langle r_w^* \rangle} \otimes \mathbb{Z}[1/\ell],$$

$$K^*\big(M; \mathbb{Z}[1/\ell]\big) \cong K^*(S^1) \otimes K^*(G/H)^{\langle r_w^* \rangle} \otimes \mathbb{Z}[1/\ell],$$

respectively, where $(-)^{\langle w^* \rangle}$ denotes the subring of w^*-invariant elements, the $K^*(S^1)$-module structure is given in both cases by pullback from $M/G \approx S^1$, and the RG-algebra structure is induced by the inclusion $H \longhookrightarrow G$.

[6] Explicitly, naturality of multiplication implies the suspension isomorphism $\widetilde{E}^*(Y_+) \longrightarrow \widetilde{E}^{*+1}(S^1 \wedge Y_+)$ is given by multiplication by the pullback of z, giving a natural nonunital ring isomorphism $\widetilde{E}^*S^1 \otimes_{E^*} \widetilde{E}^*(Y_+) \longrightarrow \widetilde{E}^*(S^1 \wedge Y_+)$. From the cofiber sequence $S^1 \vee Y_+ \to S^1 \times Y_+ \to S^1 \wedge Y_+$ we get $E^*S^1 \otimes_{E^*} E^*(Y_+) \xrightarrow{\sim} E^*(S^1 \times Y_+)$ and from $Y \to Y_+ \leftrightarrows *$ we get $E^*S^1 \otimes_{E^*} E^*Y \xrightarrow{\sim} E^*(S^1 \times Y)$.

Remark 1.8. There is a transfer map in K-theory we could also apply directly to bypass this level of generality.

Such a clean statement is not possible without inverting the order of w.

Example 1.9. Let $G = \mathrm{SO}(n)$ and K the block-diagonal subgroup $[1]^{\oplus n-2} \oplus \mathrm{SO}(2)$. Then $N_G(K)$ has two components, represented by the identity matrix and the block-diagonal $w = [1]^{\oplus n-3} \oplus [-1] \oplus \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, conjugation by which corresponds to complex conjugation under the standard identification of $\mathrm{U}(1)$ with the unit circle in the complex plane. Thus w acts on $R\mathrm{SO}(2) \cong \mathbb{Z}[t^{\pm 1}]$ by $t \leftrightarrow t^{-1}$, where $t\colon \mathrm{SO}(2) \xrightarrow{\sim} \mathrm{U}(1)$ is the defining representation on $\mathbb{C} \cong \mathbb{R}^2$. We let M be the mapping torus of the right action of w on G/K. To proceed integrally rather than via Proposition 1.7, we use the Mayer–Vietoris sequence of the cover of M by two intervals overlapping at the endpoints. This is an exact sequence

$$0 \to K^0_G M \longrightarrow RK \times RK \longrightarrow RK \times RK \longrightarrow K^1_G M \to 0$$

where the middle map is $(a, b) \longmapsto (a - b, a - wb)$. Since the first map is diagonal, the middle map may be replaced with the map $\phi\colon RK \longrightarrow RK$ taking a to $a - wa$. Thus

$$K^0_G(M) \cong \ker\phi = R(K)^{\langle w \rangle} = \mathbb{Z}[t + t^{-1}],$$

$$K^1_G(M) \cong \operatorname{coker}\phi = \mathbb{Z}[t^{\pm 1}] \big/ \mathbb{Z}\{t^n - t^{-n} : n \in \mathbb{N}\}.$$

Since the denominator in the cokernel induces on the numerator precisely the relations $t^{-n} \equiv t^n$, a set of coset representatives for $\operatorname{coker}\phi$ is given by $\mathbb{Z}\{1, t, t^2, t^3, \ldots\}$. Writing $q = t + t^{-1}$, one sees

$$[1] \overset{q}{\longmapsto} [t] + [t^{-1}] = 2[t], \qquad [t] \overset{q}{\longmapsto} [t^2 + 1] \overset{q}{\longmapsto} [t^3 + 3t] \overset{q}{\longmapsto} [t^4 + 4t^2 + 3] \overset{q}{\longmapsto} \cdots,$$

and generally $q^n \cdot [t]$ has highest term $[t^{n+1}]$, so $K^1_G\big(M; \mathbb{Z}[1/2]\big)$ is a free cyclic $K^0_G\big(M; \mathbb{Z}[1/2]\big)$-module on $[1]$. Note that with $\mathbb{Z}$ coefficients, $K^1_G(M)$ is not a free $K^0_G(M)$-module.

2. Mayer–Vietoris and double mapping cylinders

The circle case disposed of, we begin analyzing the double mapping cylinder Figure 0.2(a) in Mostert's dichotomy 0.1 from the introduction.

The double mapping cylinder M of $\pi^\pm\colon G/H \rightrightarrows G/K^\pm$ admits an obvious invariant open cover by the respective inverse images U^- and U^+ of the subintervals $[-1, 1/2)$ and $(-1/2, 1]$ of $X/G \approx [-1, 1]$, and the intersection $W = U^- \cap U^+$ equivariantly deformation retracts to G/H and $U^\pm$ to $G/K^\pm$ in such a way that the inclusions $W \longhookrightarrow U^\pm$ correspond to the projections $\pi^\pm$. Since $K^*_G(G/\Gamma) = K^0_G(G/\Gamma) = R\Gamma$ for closed subgroups $\Gamma \leqslant G$ by restriction of an equivariant bundle to the identity coset $1\Gamma \in G/\Gamma$ and K^*_G is $\mathbb{Z}/2$-graded [Seg68, Ex. (ii), p. 132; Prop. (3.5)], the Mayer–Vietoris sequence in K-theory reduces to the exact sequence

$$0 \to K^0_G(M) \longrightarrow RK^- \times RK^+ \longrightarrow RH \xrightarrow{\delta} K^1_G(M) \to 0$$

noted in the introduction. As promised there, this sequence is more informative than one might expect, reflecting the fact that in great generality, the properties of the Mayer–Vietoris sequence are better than is commonly acknowledged. Those who do not care about generality can safely substitute $E^* = K^*_G$ everywhere in the following without loss.

Proposition 2.1. *Let E^* be a multiplicative ($\mathbb{Z}$-graded, G-equivariant) cohomology theory. Then the natural $E^*(X)$-module structure on the terms of the Mayer–Vietoris sequence of a triad $(X; U, V)$ of G–CW complexes with $X = U \cup V$ is preserved by the connecting map in the sequence.*

The additional structure on the connecting map is most helpful when even or odd cohomology of the constituent subsets vanishes, making the connecting map surjective.

Example 2.2. Let M be a closed, oriented 3-manifold. Then M can be triangulated. A regular neighborhood U of its 1-skeleton is an open handlebody (i.e., homeomorphic to the bounded component cut out of $\mathbb{R}^3$ by an embedded closed surface), and examining the local picture in each 3-simplex, one sees the interior of the complement V is also a handlebody. The closures of U and V meet in a closed, oriented surface S_g, and this assemblage is called a *Heegaard splitting* of M. Letting N_g denote a standard genus-g handlebody with boundary S_g, we may write $M \approx N_g \cup_f N_g$ for some gluing homeomorphism $f \colon S_g \longrightarrow S_g$. If we write a_j for the standard g circles generating $H_1(N_g)$ and b_j for the g circles bounding discs in S_g representing the other standard generators, so that $|a_i \cap b_j| = \delta^i_j$, then M is determined up to homeomorphism by the images $f(b_j)$. Let α_j and β_j be the dual basis of $H^1(S_g)$.

Fattening U and V slightly, we may apply the Mayer–Vietoris sequence in cohomology, which contains the subsequence

$$0 \to H^1(M) \xrightarrow{\varkappa} \mathbb{Z}^g \oplus \mathbb{Z}^g \xrightarrow{\lambda} H^1(S_g) \xrightarrow{\delta} H^2(M) \to 0.$$

Thus $H^1(M)$ and $H^2(M)$ are determined by the map λ, which is in turn determined by the map f. If we make the identifications $U \cap V = S_g \subsetneq N_g = U$, then the first component $\lambda_1 \colon \mathbb{Z}^g \longrightarrow H^1(S_g)$ is the inclusion $\iota^* \colon \alpha_j \longmapsto \alpha_j$ and the second component λ_2 is $f^*\iota^*$, so we have an isomorphism

$$\operatorname{im}\delta \cong \operatorname{coker}\lambda = \frac{\mathbb{Z}\{\alpha_j, \beta_j\}}{\mathbb{Z}\{\alpha_j, f^*\alpha_j\}},$$

which in particular is spanned by the images of the β_j, and $H^1(M) \cong \ker\lambda$ is spanned by elements $(\sum m_i\alpha_i, \sum n_j\alpha_j)$ such that $\sum n_j f^*\alpha_j$ has no β-component. By Proposition 2.1, the cup product $\mu_{1,2} \colon H^1(M) \times H^2(M) \longrightarrow H^3(M)$ is determined by $y \smile \delta(z) = \delta(\lambda_1\varkappa y \smile z)$, where $\lambda_1\varkappa y$ is some linear combination of the α_i and z can be taken to be a linear combination of the β_j, and the second cup product is taken in $H^*(S_g)$. Since this product is given on generators by $\alpha_i \smile \beta_j = \delta^i_j$, the Mayer–Vietoris sequence gives $\mu_{1,2}$ in terms of $H_1(f)$.

Though Proposition 2.1 does not seem to appear as such in the literature, with a bit of faith it is possible to cobble together a proof from citations.

Terse proof of Proposition 2.1. In the long exact sequence of a pair (X, A), the connecting map $E^*(A) \longrightarrow E^{*+1}(X, A)$ is an $E^*(X)$-module homomorphism; see Whitehead [Whi62, (6.19), p. 263] for an algebraic proof for cohomology theories represented by ring spectra and note the proof still follows from our axioms. Up to homotopy, the Mayer–Vietoris sequence of $(X; U, V)$ is the long exact sequence of a pair $(X', U' \amalg V')$ in which X' is homotopy equivalent to X via a homotopy equivalence $X' \longrightarrow X$ sending disjoint G–CW subcomplexes U' and V' respectively to U and V; *cf.* Adams [Adams74, p. 213] for a version of this statement for a representable theory.[7] □

[7] Another version of this statement appears in a MathOverflow solution due to J. Peter May [May] for CW-spectra (or, to quote, "any halfway reasonable category" of spectra).

This is in a moral sense a geometry paper, so for those with less faith, a more expansive and geometric account follows.

Notation 2.3. In what follows between now and the return to K-theory, all maps will be equivariant with respect to a fixed topological group G and all G-spaces will come equipped with a G-fixed basepoint $*$. The wedge sum and smash product inherit the expected actions, and the closed unit interval $I = [0,1]$ and circle $S^1 = I/(0 \sim 1)$ are basepointed at 0 and equipped with the trivial G-action. We write $CX = I \wedge X$ for the reduced cone and $\Sigma X = CX/X = S^1 \wedge X$ for the reduced suspension, with the induced actions.

The G-structure is just along for the ride in the proof that follows, and everything we state through to Proposition 2.9 follows for nonequivariant theories through the expedient of setting $G = 1$.

Definition 2.4. Let $\widetilde{E}^*$ be a multiplicative G-equivariant cohomology theory (not even necessarily equipped with suspension maps). The diagonal $\Delta\colon X \longrightarrow X \wedge X$ makes a G-space X a *coalgebra* in the sense that $(\Delta \wedge \mathrm{id}) \circ \Delta = (\mathrm{id} \wedge \Delta) \circ \Delta$. A right *$X$-coaction* $\Delta_Y\colon Y \longrightarrow Y \wedge X$ on a G-space Y is a map such that $(\Delta_Y \wedge \mathrm{id}) \circ \Delta_Y = (\mathrm{id} \wedge \Delta) \circ \Delta_Y$; such a map makes Y a right *X-comodule* and induces an additive homomorphism $\Delta_Y \circ \mu_{Y,X}\colon \widetilde{E}^*Y \otimes \widetilde{E}^*X \longrightarrow \widetilde{E}^*Y$ which one checks, unravelling definitions, to be a right $\widetilde{E}^*X$-algebra structure. A map $f\colon Y \longrightarrow Z$ between right X-comodules such that $\Delta_Z \circ f = (f \wedge \mathrm{id}) \circ \Delta_Y$ is an *X-comodule homomorphism*, and induces a $\widetilde{E}^*X$-algebra homomorphism $f^*\colon \widetilde{E}^*Z \longrightarrow \widetilde{E}^*Y$.

Proposition 2.5. *Let G be a topological group and E^* a multiplicative G-equivariant cohomology theory. Then in the long exact sequence of a G–CW pair (X, A), all objects are E^*X-modules and all arrows E^*X-module homomorphisms. In particular the image of $E^*(X/A) \longrightarrow E^*X$ is an ideal and the image of $E^*A \longrightarrow \widetilde{E}^{*+1}(X/A)$ is a nonunital subring with zero multiplication.*

We adapt a proof from Hatcher's manuscript K-theory text [HatVBKT, Prop. 2.15], which considers the cross product with a single element and does not make explicit use of the notion of a comodule.

Proof. It will be enough to prove the result for the reduced theory $\widetilde{E}^*$. Note that for pointed G–CW subcomplexes A of X and pointed G–CW complexes S with trivial action, $S \wedge A$ admits the X-coaction $s \wedge a \longmapsto s \wedge a \wedge a$ and $S \wedge (X \cup CA)$ the X-coaction

$$\begin{aligned} s \wedge x &\longmapsto s \wedge x \wedge x, \\ s \wedge t \wedge a &\longmapsto s \wedge t \wedge a \wedge a. \end{aligned}$$

It is easy to check these coactions make a cofiber sequence $A \to X \to X \cup CA$ a sequence of X-comodule homomorphisms. To see this also makes the Puppe sequence

$$A \xrightarrow{i} X \longrightarrow X \cup CA \longrightarrow \Sigma A \xrightarrow{\Sigma i} \Sigma X \longrightarrow \Sigma(X \cup CA) \longrightarrow \Sigma^2 A \longrightarrow \cdots$$

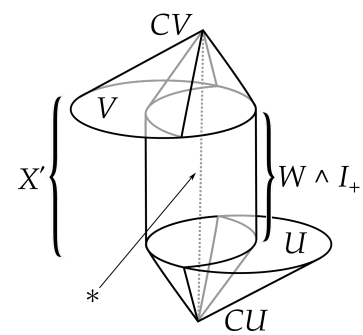
a sequence of X-comodule homomorphisms, it suffices to observe the coaction commutes with (suspensions of) the connecting map $X \cup CA \longrightarrow S^1 \wedge A$ given by $t \wedge a \longmapsto (1-t) \wedge a$ and $x \longmapsto *$. To replace $S \wedge (X \cup CA)$ with $S \wedge X/A$, observe the coaction $s \wedge [x] \longmapsto s \wedge [x] \wedge x$ on the latter makes the collapse map another X-comodule homomorphism.

Applying $\widetilde{E}^*$ to the Puppe sequence then yields an $\widetilde{E}^*X$-module structure on the long exact sequence of (X, A). To see the image of the connecting map has trivial multiplication, note this map can be written as $\widetilde{E}^*\Sigma A \longrightarrow \widetilde{E}^*(X/A)$. ∎

Remark 2.6. The meticulous reader will observe that the proof of Proposition 2.5 makes use of the fact the coaction smashes with X on one side and the suspension smashes with S^1 on the other. This choice actually matters; the choice of a left E^*X-action instead of a right requires an additional sign, making the connecting map fail to be an E^*X-module homomorphism.[8] One could be forgiven for suspecting this has something to do with the well-known sign in the Puppe sequence: our choice of $q\colon t \wedge a \longmapsto (1-t) \wedge a$ for the map $X \cup CA \longrightarrow A \wedge S^1$ comes from a nonstandard identification $CX \cup CA \twoheadrightarrow \Sigma A \xrightarrow{-} \Sigma A$ in transitioning from the iterated cofiber sequence to the Puppe sequence. This choice of identification makes E^*q the *opposite* $-\delta$ of the connecting map $\delta\colon E^*A \longrightarrow E^{*+1}(X, A)$ defined through the axioms but makes the next map ΣE^*i rather than the $-\Sigma E^*i$ it would become under the standard identification. As q and its variant $-q$ are both X-comodule maps, the choice between them is immaterial to the success of Proposition 2.5, and moreover, this choice inflicts a global sign of -1 on the connecting maps in each degree, so the correction factor arising from putting the E^*X-action on the left would be a separate, logically independent sign.

To obtain the same result on connecting maps for the Mayer–Vietoris sequence, we realize it as the long exact sequence of a pair, as in the terse proof.

Figure 2.7: Schematic of $CU \cup X' \cup CV$ in Proposition 2.8



Proposition 2.8. *Let $(X; U, V)$ be a triad of G–CW complexes with $X = U \cup V$. Write W for the intersection $U \cap V$ and X' for the double mapping cylinder $\big(U \times \{0\}\big) \cup (W \times I) \cup \big(V \times \{1\}\big)$ of the inclusions $U \hookleftarrow W \hookrightarrow V$. Then for any G-equivariant cohomology theory, the long exact sequence of the pair $\big(X', U \times \{0\} \amalg V \times \{1\}\big)$ is the Mayer–Vietoris sequence of the triad $(X; U, V)$.*

Proof. It is again enough to assume W is pointed and prove the result for the reduced theory. In so doing, we replace $W \times I$ with the reduced cylinder $W \wedge I_+ = (W \times I)/\big(\{*\} \times I\big)$, turning X' into $X'' = X'/\big(\{*\} \times I\big)$ and $U \times \{0\} \amalg V \times \{1\}$ into $U \vee V$, which is naturally a subspace of X'' since the basepoints $(*, 0)$ and $(*, 1)$ have been identified. The result is as in Figure 2.7.

[8] In detail, for singular cohomology, the k-submodule $C^*(X, A; k)$ of cochains vanishing on $C_*(A)$ is a two-sided ideal of $C^*(X; k)$ with respect to the cup product, which thus restricts to both a right and a left action of $C^*(X; k)$ on $C^*(X, A; k)$. Using the zig-zag lemma to compute the connecting map δ of the short exact sequence $C^*(X, A; k) \to C^*(X; k) \to C^*(A; k)$ of cochain complexes gives $\delta\big(a \smile i^*(x)\big) = \delta a \cdot x$ but $\delta\big(i^*(x) \smile a\big) = (-1)^{|x|} x \cdot \delta a$. In terms of our preceding discussion, the sign arises because the connecting map of the pair (X, A) factors as the composition of ring homomorphisms and the suspension isomorphism $H^*(A; k) \xrightarrow{\delta} H^{*+1}(CA, A) \xleftarrow{\sim} \widetilde{H}^{*+1}(\Sigma A)$ arising from the long exact sequence of the pair (CA, A) and the standard homeomorphism $CA/A \approx \Sigma A$; but since the suspension isomorphism can be identified as $H^*(A; k) \xrightarrow{\sim} H^1(S^1; k) \otimes_k H^*(A; k) \xrightarrow{\times} \widetilde{H}^{*+1}(S^1 \wedge A)$, the cross product *on the left* with the fundamental class of S^1, a sign can be avoided only by switching the side on which $H^*(X; k)$ acts.

Note X'' is G–homotopy equivalent to X via the map collapsing the I-direction in the reduced cylinder $W \wedge I_+$. The Puppe sequence begins

$$U \vee V \longrightarrow X'' \hookrightarrow CU \cup X'' \cup CV \xrightarrow{/X''} \Sigma U \vee \Sigma V \longrightarrow \Sigma X''.$$

We can replace the third term with ΣW because the map collapsing $CU \vee CV$ to a point is a G–homotopy equivalence. The maps then yield an exact sequence of graded groups

$$\widetilde{E}^* U \oplus \widetilde{E}^* V \longleftarrow \widetilde{E}^* X \xleftarrow{\delta} \widetilde{E}^{*-1} W \xleftarrow{\zeta} \widetilde{E}^{*-1} U \oplus \widetilde{E}^{*-1} V \longleftarrow \widetilde{E}^{*-1} X,$$

which we check is the Mayer–Vietoris sequence:

- That $U \vee V \hookrightarrow X''$ yields the pair of restrictions $\widetilde{E}^* X \longrightarrow \widetilde{E}^* U \oplus \widetilde{E}^* V$ is clear.
- The connecting map in the Mayer–Vietoris sequence is defined as the composition

$$\widetilde{E}^{*-1} W \longrightarrow \widetilde{E}^*(V/W) \xleftarrow{\sim} \widetilde{E}^*(X/U) \longrightarrow \widetilde{E}^* X,$$

  where the first map is the connecting map in the long exact sequence of the pair (V, W), hence induced by $V/W \xleftarrow{\sim} V \cup CW \to \Sigma W$, the second is the excision arising from the homeomorphism $V/W \longrightarrow X/U$, and the last is induced by the projection $X \twoheadrightarrow X/U$. Thus the Mayer–Vietoris connecting map is obtained by following the path from X to ΣW along the bottom of the following commutative diagram, while δ comes from following along the top:

$$\begin{array}{ccccc} \Sigma W & \xleftarrow{/U \vee V} & X'' & \xrightarrow{\sim} & X \\ \uparrow{\scriptstyle /V} & \swarrow{\scriptstyle /U} & & & \downarrow \\ V \cup CW & \xrightarrow[/CW]{\sim} & V/W & \xrightarrow{\approx} & X/U. \end{array}$$

- The map ζ is induced as the composition along the right in the commutative diagram

$$\begin{array}{ccccc} \Sigma W \xrightarrow{\approx} & CW \cup (W \wedge I_+) \cup CW & \hookrightarrow & CU \cup (W \wedge I_+) \cup CV \\ & \downarrow{\scriptstyle /(W \wedge I_+)} & & \downarrow{\scriptstyle /(W \wedge I_+)} \\ & \Sigma W \vee \Sigma W & \hookrightarrow & \Sigma U \vee \Sigma V. \end{array}$$

  On the other hand, the left vertical map collapsing a cylinder's worth of Ws is G–homotopy equivalent to the pinch map $\Sigma W \twoheadrightarrow \Sigma W \vee \Sigma W$ collapsing only the equator $W \times \{1/2\}$, so the composition $\Sigma W \to \Sigma W \vee \Sigma W \to \Sigma U \vee \Sigma V$ is homotopic to $\Sigma j_U - \Sigma j_V$, where $j_U, j_V \colon W \hookrightarrow U, V$ are the inclusions. The minus sign comes from observing a small neighborhood the cone point of the abstract $CU = U \wedge I$ lies near suspension coordinate $t = 0$, agreeing with the suspension coordinate of the included copy of CU in $CU \cup (W \wedge I_+) \cup CV$, while the cone point of the included copy of CV is near $t = 1$, disagreeing with that of the abstract CV. □

The conjunction of these two results gives Proposition 2.1. Taking $W = U \cap V$ in the statement, the image of $\delta\colon E^{*-1}W \longrightarrow E^*X$ is an ideal with multiplication zero, since δ is induced by $X \longrightarrow \Sigma W$ and the multiplication of the non-unital algebra $\widetilde{E}^*\Sigma W$ is zero. This result allows us to completely compute the ring E^*X from E^*U, E^*V, and E^*W in amenable cases. We write $j_U, j_V\colon W \hookrightarrow U, V$ and $i_U, i_V\colon U, V \hookrightarrow X$.

Proposition 2.9. *Let E^* be a $\mathbb{Z}$-graded G-equivariant multiplicative cohomology theory and $(X; U, V)$ a triple of G–CW complexes with $X = U \cup V$ and such that the odd-dimensional E-cohomology of U, V, and $W = U \cap V$ vanishes. Then one has a graded ring and a graded E^*W-module isomorphism, respectively:*

$$E^{\mathrm{even}}X \cong E^*U \underset{E^*W}{\times} E^*V, \qquad E^{\mathrm{odd}}X \cong \Big(E^*W \big/ \operatorname{im} j_U^* + \operatorname{im} j_V^*\Big)[1].$$

The multiplication of odd-degree elements is zero, and the product $(x, \delta w) \in E^{\mathrm{even}}X \times E^{\mathrm{odd}}X \longrightarrow E^{\mathrm{odd}}X$ descends from the multiplication of E^*W in the sense that $x \cdot \delta w = \delta\big(j_U^* i_U^*(x) \cdot w\big)$.

Proof. The additive isomorphisms follow from the reduction of the Mayer–Vietoris sequence to

$$0 \to E^{\mathrm{even}}X \xrightarrow{\ i\ } E^*U \times E^*V \longrightarrow E^*W \xrightarrow{\ \delta\ } E^{\mathrm{odd}}X \xrightarrow{i} 0.$$

The multiplication in the even subring follows because i is the ring homomorphism induced by $U \amalg V \longrightarrow X$. The product of odd-degree elements $x, y \in E^{\mathrm{odd}}X$ is zero by Proposition 2.1 since δ is surjective.[9] To multiply an even-degree element x with an odd-degree element δw, note that δ is an E^*X-module homomorphism by Proposition 2.1, so particularly $x \cdot \delta w = \delta(x \cdot w)$. Now recall the module structure on E^*W is given by restriction as $x \cdot w = (i_U \circ j_U)^*(x) \cdot w$. □

Remark 2.10. In this paper, of course, we take $E^* = K_G^*$. In our previous joint work [CGHM19], we took E^* to be Borel cohomology $X \longmapsto H\mathbb{Q}^*(EG \otimes_G X)$, so that $E^*(G/\Gamma) = H\mathbb{Q}^* B\Gamma$ is concentrated in even degree by Borel's theorem; generally, given a nonequivariant cohomology theory e^* such that $e^*(*)$ is torsion in odd degrees, one could rationalize and take E^* to be rational Borel G-equivariant e-cohomology $e\mathbb{Q}_G^*$ so that $E^n(G/\Gamma) = e\mathbb{Q}^n B\Gamma$. Since we have rationalized [Rud98, Cor. 7.12], the Atiyah–Hirzebruch spectral sequences of CW-skeleta $B_n\Gamma$ collapse at $E_2 = H^*(B_n\Gamma; \mathbb{Q}) \otimes e^*(*)$, which is concentrated in even degree, so that $E^*(G/\Gamma) = e\mathbb{Q}_G^* B\Gamma$ is concentrated in even degree as well and Proposition 2.9 applies. The author is unsure how much demand there is for $e\mathbb{Q}_G^*$, but has at least sighted the "Borel equivariant complex bordism" functor $X \longmapsto MU_*(EG \otimes_G X)$ in the wild.

We can now finally return to K-theory.

Theorem 2.11. *Let M be the double mapping cylinder of the projections $\pi^\pm\colon G/H \rightrightarrows G/K^\pm$. The Mayer–Vietoris sequence reduces to a short exact sequence*

$$0 \to K_G^0 M \longrightarrow RK^- \times RK^+ \longrightarrow RH \longrightarrow K_G^1 M \to 0$$

of $K_G^0 M$-module homomorphisms, inducing the following graded ring and graded RH-module isomorphism, respectively:

$$K_G^0(X) \cong RK^- \underset{RH}{\times} RK^+, \qquad K_G^1(X) \cong \Big(RH \big/ RK^-|_H + RK^+|_H\Big)[1],$$

[9] Alternatively, since i is injective on $E^{\mathrm{even}}X$ and vanishes on $E^{\mathrm{odd}}X$, we have $i(xy) = ix \cdot iy = 0$ so $xy = 0$.

where $(-)|_H$ denotes restriction of representations along the inclusions $H \hookrightarrow K^{\pm}$. The product of odd-degree elements is zero, and the product $K^0_G(X) \times K^1_G(X) \longrightarrow K^1_G(X)$ descends from the multiplication of RH:

$$(\rho_-, \rho_+) \cdot \overline{\sigma} = \overline{\rho_-|_H \cdot \sigma}$$

for (ρ_-, ρ_+) in the fiber product $RK^- \underset{RH}{\times} RK^+$ and $\overline{\sigma} \in K^1_G(X)$ the image of $\sigma \in RH$.

Example 2.12. Let $G = \mathrm{O}(n)$ with $K = K^{\pm} = \mathrm{O}(3)$ and $H = \mathrm{O}(2)$ block-diagonal. Recall that $R\mathrm{O}(3) \cong R\mathrm{SO}(3) \times R(\mathbb{Z}/2) = \mathbb{Z}[\sigma, \varepsilon]/(\varepsilon^2 - 1)$, where $\sigma\colon \mathrm{O}(3) \hookrightarrow \operatorname{Aut}\mathbb{R}^3 \to \operatorname{Aut}\mathbb{C}^3$ complexifies the defining representation and $\varepsilon = \det\colon \mathrm{O}(3) \longrightarrow \operatorname{Aut}\mathbb{C}$ is the determinant, and $R\mathrm{O}(2) \cong \mathbb{Z}[\rho, \varepsilon]/(\varepsilon^2 - 1, \rho\varepsilon - \rho)$, where $\rho\colon \mathrm{O}(2) \longrightarrow \operatorname{Aut}\mathbb{C}^2$ complexifies the defining representation [Min71]. The restriction $RK \longrightarrow RH$ is given by $\sigma \mapsto \rho + 1$ and $\varepsilon \mapsto \varepsilon$. Now Theorem 2.11 yields a short exact sequence

$$0 \to K^0_G(M) \longrightarrow \frac{\mathbb{Z}[\sigma,\varepsilon]}{(\varepsilon^2 - 1)} \times \frac{\mathbb{Z}[\sigma,\varepsilon]}{(\varepsilon^2 - 1)} \longrightarrow \frac{\mathbb{Z}[\rho,\varepsilon]}{(\varepsilon^2 - 1, \rho\varepsilon - \rho)} \to 0.$$

The kernel decomposes additively as the sum

$$K^*_G(M) = K^0_G(M) \cong \left\{ (x,x) : x \in \frac{\mathbb{Z}[\sigma,\varepsilon]}{(\varepsilon^2 - 1)} \right\} \oplus \big((\sigma - 1)(\varepsilon - 1), 0\big) \oplus \big(0, (\sigma - 1)(\varepsilon - 1)\big)$$

This bears a familial similarity to the description in Theorem 0.4(b) but cannot be put in those terms due to torsion.

The cohomological situation, by way of contrast, is much simpler: we have $H^*_K \cong \mathbb{Q}[p_1] \cong H^*_H$, where p_1 the first Pontrjagin class of the tautological bundle over the infinite Grassmannian $\mathrm{Gr}(3, \mathbb{R}^\infty) = B\mathrm{O}(3)$, so $H^*_G M \cong \mathbb{Q}[p_1]$. The equivariant Chern character taking a representation V to the Chern character of the associated vector bundle $V_{\mathrm{O}(3)} \to B\mathrm{O}(3)$ sends $\sigma - 3$ to p_1 and annihilates $\varepsilon - 1$.

Example 2.13. If $G = K^{\pm} = H$, the resulting double mapping cylinder is just the unreduced suspension $S(G/H)$ and one has

$$K^0_G\big(S(G/H)\big) = RG, \qquad K^1_G\big(S(G/H)\big) = {}^{RH}\!/_{\mathrm{im}(RG \to RH)}\,[1].$$

Remark 2.14. The decomposition in Theorem 2.11 admits a winning interpretation in terms of bundles. The isomorphism $RK^- \times_{RH} RK^+ \xrightarrow{\sim} K^0_G(M)$ comes explicitly from the decomposition of the double mapping cylinder as the union along G/H of the mapping cylinders $M(G/H \to G/K^{\pm})$ of the natural quotient maps $G/H \longrightarrow G/K^{\pm}$ for any pair $\sigma^{\pm}$ of $K^{\pm}$-representations agreeing on H, one forms the union of the bundles $M(G \otimes_H V_{\sigma^\pm} \to G \otimes_{K^\pm} V_{\sigma^\pm}) \longrightarrow M(G/H \to G/K^{\pm})$ along the restriction $G \otimes_H V_{\sigma^\pm} \longrightarrow G/H$ to their common boundary. Particularly, for a K^+-representation σ^+ which is trivial on H, one can extend the bundle $M(G \otimes_H V_{\sigma^+} \to G \otimes_{K^+} V_{\sigma^+})$ by gluing on a trivial bundle over $M(G/H \to G/K^-)$; call this ξ_{σ^+}. The formal difference of $\xi^+_{\sigma^+}$ and the trivial bundle $\underline{\mathbb{C}}^{\dim V_{\sigma^+}}$ is a typical element of the summands $\overline{\rho}RH[\overline{\rho}]$ and $(t-1)RH[t^{\pm 1}]$ figuring in Theorem 4.1(a).

For Theorem 4.1(b), one similarly forms a virtual bundle ξ_{σ^-} from a K^--representation σ^- trivial on H. That the product $(\xi_{\sigma^-} - \underline{\mathbb{C}}^{\dim V_{\sigma^-}}) \otimes (\xi_{\sigma^+} - \underline{\mathbb{C}}^{\dim V_{\sigma^+}})$ should be zero follows by noting the first factor is zero over $M(G/H \to G/K^-)$ and the second over $M(G/H \to G/K^+)$.

The map $RH \longrightarrow K^1_G(M)$ admits the following description. Given an H-representation σ, use Bott periodicity to send the class of the bundle $G \otimes_H V_\sigma$ to an element of $K^0_G\big(S^2(G/H)\big)$, and then pull back to an element of $K^0_G(SM)$ along the suspension of the map $M \twoheadrightarrow S(G/H)$ collapsing each of the end-caps $G/K^\pm$ to a point.

Hodgkin [Hodgkin, Cor. 10.1] notes the geometric significance of the class $\overline{\beta}(\rho) \in K^1(K/H)$, for ρ a K-representation trivial on H, is as the class of the bundle on $S(K/H)$ obtained by gluing trivial bundles $\underline{V}_\rho$ over two copies of the cone $C(K/H)$ along their boundaries K/H via the identification $(kH, v) \sim \big(kH, \rho(k)v\big)$.

3. Restrictions of representation rings

To say anything more meaningful about the map $RK^- \times RK^+ \longrightarrow RH$ figuring in Theorem 2.11, unsurprisingly, we will have to do some representation theory.

Definition 3.1. If Γ is any group, we write Γ' for its commutator subgroup and Γ^{ab} for its abelianization. We then have a functorial short exact sequence $1 \to \Gamma' \to \Gamma \to \Gamma^{\mathrm{ab}} \to 1$. The center of Γ is denoted by $Z(\Gamma)$ and the connected component of the identity element by Γ_0. If two groups Π and A contain a subgroup F central in both, we write $\Pi \otimes_F A$ for the balanced product $(\Pi \times A)/\big\{(f, f^{-1}) : f \in F\big\}$. When a group Γ is isomorphic to such a balanced product with F finite, we refer to the isomorphism as a ***virtual product decomposition***. It is well known that a compact, connected Lie group Γ admits a virtual product decomposition $\Gamma \cong \Gamma' \underset{F}{\otimes} Z(\Gamma)_0$, and F is the intersection of Γ' and $Z(\Gamma)_0$.

A representation ring $R\Gamma$ is augmented over $\mathbb{Z}$ by the unique $\mathbb{Z}$-linear map taking an honest representation to its dimension. Given a commutative ring k, the category of augmentation-preserving maps of augmented k-algebras is *pointed* in the sense it admits k as a zero object. The kernel of the augmentation $A \longrightarrow k$ is denoted IA, or, if $A = R\Gamma$ is a representation ring, $I\Gamma$. The quotient k-module $IA/(IA)^2$, the ***module of indecomposables***, is written QA. Specializing the general definition of exactness in a pointed category, a sequence of augmented k-algebras $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be ***exact*** at B if $\ker g = f(IA)B$. A short exact sequence $k \to A \longrightarrow B \longrightarrow C \to k$ of augmented k-algebras is said to be ***split*** if there exists a section $C \rightarrowtail B$ inducing an isomorphism $A \otimes_k C \xrightarrow{\sim} B$.

Given an inclusion $A \hookrightarrow B$ of rings, an element $b \in B$ is said to be ***transcendental*** over A if the A-algebra map $A[x] \longrightarrow B$ from the polynomial ring in one indeterminate over A sending x to b is injective.

3.1. The splitting lemma

We need a refinement of the following splitting lemma due to Hodgkin.

Theorem 3.2 ([Hodgkin, Prop. 11.1])**.** *Given any compact, connected Lie group K with free abelian fundamental group, the sequence*

$$\mathbb{Z} \to RK^{\mathrm{ab}} \longrightarrow RK \longrightarrow RK' \to \mathbb{Z}$$

induced by abelianization is split exact.

This essentially allows us to factor out the representation ring of the connected component of the center of a Lie group. We actually want to factor out an arbitrary central torus. In order for this to work we need RK' to be a polynomial ring, or equivalently, that K' be a direct product of simply-connected groups and odd special orthogonal groups [Ste75].[10]

Proposition 3.3. *Let K be a compact, connected Lie group such that RK' is a polynomial ring and let $\underline{K}$ be a connected subgroup containing K' with free abelian fundamental group and A a virtual complement, meaning a central torus with $F = \underline{K} \cap A$ finite and such that $K \cong \underline{K} \otimes_F A$. Then the sequence*

$$\mathbb{Z} \to R(A/F) \longrightarrow RK \longrightarrow R\underline{K} \to \mathbb{Z} \tag{3.4}$$

induced by the exact sequence $1 \to \underline{K} \to K \to A/F \to 1$ is split exact. The splitting is not natural.

Proof. Hodgkin already proved the statement in the first paragraph in the case where $\underline{K} = K'$ is simply-connected. His argument in fact only uses that RK' is polynomial, so we can get away assuming only this. Then Hodgkin's argument is obtained from the one below by letting $(\underline{K}, K')$ respectively take the roles of $(K, \underline{K})$. This shows $R\underline{K}$ is the tensor product of a polynomial and a Laurent algebra, by the split exactness of (3.4), so that the argument now applies in general to give a splitting of RK in terms of $R\underline{K}$.

The argument. The restriction $\underline{K} \times A \longrightarrow K$ of the multiplication of K is a surjective homomorphism with kernel the antidiagonal $\nu F = \{(f, f^{-1}) : f \in F\}$ inducing the evident isomorphism $\underline{K} \otimes_F A \xrightarrow{\sim} K$. Pulling back, representations of K can be identified with those representations of $\underline{K} \times A$ whose kernels contain νF. The projections $\underline{K} \times A \twoheadrightarrow A \twoheadrightarrow A/F$ give us the first map $R(A/F) \rightarrowtail R(\underline{K} \times A)$ in the display.

For the second map, it will suffice to lift a list (ρ_j) of representations of $\underline{K}$ forming a minimal set of polynomial and Laurent generators for $R\underline{K}$, making sure the lifts of the Laurent generators are still units. To lift an irreducible $\rho \colon \underline{K} \longrightarrow \operatorname{Aut} \mathbb{C}^n$ to a representation of $\underline{K} \times A$ trivial on νF, note that since F is central, multiplication by each element of $\rho(F)$ is a $\underline{K}$-module endomorphism of $\mathbb{C}^n$, and hence by Schur's lemma, a constant times $\mathrm{id}_{\mathbb{C}^n}$, so $\rho|_F$ is a direct sum of n copies of some one-dimensional representation $\bar{\sigma} \colon F \longrightarrow S^1$. Since $\operatorname{Hom}(-, S^1)$ is exact and F a subset of A, taking $\rho = \rho_j$, we see $\bar{\sigma}$ is the restriction of some $\sigma_{\rho_j} \colon A \longrightarrow S^1$. For each j, consider the representation $\widetilde{\rho}_j := \rho_j \otimes \sigma_{\rho_j}$ of $\underline{K} \times A$ in $\mathbb{C}^n$ taking $(k, a) \longmapsto \sigma_{\rho_j}(a)\,\mathrm{id}_{\mathbb{C}^n} \cdot \rho_j(k)$. This $\widetilde{\rho}_j$ vanishes on νF by construction and restricts to ρ_j on $\underline{K}$. In case $\rho_j \colon \underline{K} \longrightarrow S^1$ was one of the Laurent generators, then $n = 1$, so $\widetilde{\rho}_j$ is still a one-dimensional representation and hence invertible.

It remains to show the map is an isomorphism. We have maps

$$R\underline{K} \otimes R(A/F) \xrightarrow{\phi} RK \rightarrowtail R\underline{K} \otimes RA,$$

where ϕ is defined in the expected manner from the maps we have just constructed and the second map comes from the covering $\underline{K} \times A \longrightarrow K$ and the natural identification $R(\underline{K} \times A) \cong R\underline{K} \otimes RA$. Since $A \longrightarrow A/F$ is surjective, $\operatorname{Hom}(A/F, S^1) \longrightarrow \operatorname{Hom}(A, S^1)$ and hence $R(A/F) \longrightarrow RA$ are injective. Hence the composition is injective on elements of the form $p(\vec{\rho}) \otimes \theta$, where $p(\vec{\rho})$ is a Laurent monomial in the generators ρ_j and θ is an element of $\operatorname{Hom}(A/F, S^1)$. As such elements form a $\mathbb{Z}$-basis for $R\underline{K} \otimes R(A/F)$, we find ϕ is injective. To see it is surjective, let any element $p(\rho_j) \otimes \theta \in R(\underline{K} \times A)$ vanishing on νF be given; such elements form a $\mathbb{Z}$-basis for the image of $RK \rightarrowtail R\underline{K} \otimes RA$. The element can be rewritten $p(\rho_j) \otimes \theta = p(\widetilde{\rho}_j) \cdot (1 \otimes \theta')$ for some

[10] The representation rings of the even special orthogonal groups and relation with those of the odd special orthogonal groups are given in (3.16)

other $\theta' \in \mathrm{Hom}(A, S^1)$. Moreover, $1 \otimes \theta' \colon (k, a) \longmapsto \theta'(a)$ is trivial on νF since $p(\rho) \otimes \theta$ and $p(\widetilde{\rho}_j)$ are, so θ' is trivial on F and hence descends to an element of $R(A/F)$. Thus $p(\rho_j) \otimes \theta = \phi\big(p(\rho_j) \otimes \theta'\big)$.

□

In the few cases we need, this unnatural splitting can actually be chosen compatibly with restrictions $R(H \hookrightarrow K)$.

Proposition 3.5. *Let K, $\underline{K}$, A, and F be as in Proposition 3.3 and let H be a closed, connected subgroup of K, also containing A, such that RH' is a polynomial ring and $\underline{H} = \underline{K} \cap H$ contains F. If the restriction $R\underline{K} \longrightarrow R\underline{H}$ is a split surjection, then a splitting of RH as in Proposition 3.3 can be chosen compatibly so that $RK \longrightarrow RH$ is identified with $R\underline{K} \otimes R(A/F) \longrightarrow R\underline{H} \otimes R(A/F)$.*

Proof. There is a natural map from $1 \to \underline{H} \to H \to A/F \to 1$ to the exact sequence for K, inducing a map of short exact sequences of representation rings. A choice of splitting $R\underline{H} \to R\underline{K}$ and the splitting $R\underline{K} \to RK$ of the first part of the proposition uniquely induces a compatible splitting $R\underline{H} \to R\underline{K} \to RK \to RH$. □

This will help us deal with the case that K/H is an odd-dimensional sphere. There is an analogous statement when K/H is an even-dimensional sphere, but to make it involves a case analysis of K and H, so it is difficult to extract it from the proof of Theorem 0.4.

Lemma 3.6. *Suppose a compact, connected Lie group K can be written as balanced product $\underline{K} \otimes_F A$ of two subgroups A and $\underline{K}$, where A is a central torus in K and F is finite, and that H is a closed subgroup of K such that K/H is a sphere S^{2n} of positive even dimension. Then, writing $\underline{H} = H \cap \underline{K}$, we have $H \cong \underline{H} \underset{F}{\otimes} A$.*

Proof. Since $\pi_1(S^{2n}) = 0$, it follows H must contain A, and it follows from the decomposition of K that $\underline{H}$ and A together generate H. The preimage of H under the projection $\underline{K} \times A \longrightarrow K$ is $F\underline{H} \times A$, so it follows $\underline{K}/F\underline{H} \approx S^{2n}$. Since $\underline{K}/\underline{H} \longrightarrow \underline{K}/F\underline{H}$ is a finite covering, we see $F\underline{H} = \underline{H}$, so $\underline{H}$ contains F. Thus one can write $H \cong \underline{H} \otimes_F A$ as claimed. □

Lemma 3.7. *Suppose a compact, connected Lie group K and closed, connected subgroup H are given such that K/H is homeomorphic to an even-dimensional sphere, and suppose K' is the direct product of a simply-connected group and some number of factors* SO(odd)*. Then there exist direct factors $\widetilde{K}_{\mathrm{eff}}$ of K' and $\widetilde{H}_{\mathrm{eff}} < \widetilde{K}_{\mathrm{eff}}$ of H' and a common direct factor L of H' and K' such that the inclusion $H' \longleftrightarrow K'$ may be identified with $\widetilde{H}_{\mathrm{eff}} \times L \longleftrightarrow \widetilde{K}_{\mathrm{eff}} \times L$ and hence the induced map $\widetilde{K}_{\mathrm{eff}}/\widetilde{H}_{\mathrm{eff}} \longrightarrow K/H$ is a diffeomorphism. The pair $(\widetilde{K}_{\mathrm{eff}}, \widetilde{H}_{\mathrm{eff}})$ is up to an isomorphism of pairs one of $\big(\mathrm{Spin}(2n+1), \mathrm{Spin}(2n)\big)$, $\big(\mathrm{SO}(2n+1), \mathrm{SO}(2n)\big)$, for $n \geqslant 1$ or $\big(G_2, \mathrm{SU}(3)\big)$.*

Proof. Note that the image K_{eff} of the action map $\alpha \colon K \longrightarrow \mathrm{Homeo}\, K/H$ is by definition effective and hence must be $\mathrm{SO}(2n+1)$ or G_2, with the image of H being $\mathrm{SO}(2n)$ or $\mathrm{SU}(3)$ respectively [Besse, Ex. 7.13][GrWZ08, Table C, p. 104]. The effective image $H_{\mathrm{eff}} = \alpha(H)$, in particular, determines K_{eff} uniquely up to isomorphism. The kernel of α contains $A = Z(K)_0$, and applying Lemma 3.6 with $\underline{K} = K'$, we have a decomposition $(K, H) = (K' \otimes_F A, \underline{H} \otimes_F A)$. Write $\underline{H} = \alpha^{-1}(H_{\mathrm{eff}}) \cap K'$; this is just H' if $H_{\mathrm{eff}} \neq \mathrm{SO}(2)$ and a virtual direct product of the form $H' \cdot S^1$ if $H_{\mathrm{eff}} = \mathrm{SO}(2)$. The inclusion of pairs $(K', \underline{H}) \longleftrightarrow (K, H)$ then induces a diffeomorphism $K'/\underline{H} \longrightarrow K/H$.

Since an element of K acting trivially on $K/H = S^{2n}$ in particular stabilizes the basepoint $1H$, we see H contains $\ker\alpha$, and similarly $\underline{H}$ contains $\ker\alpha|_{K'}$, which is thus the same as $\ker\alpha|_{\underline{H}}$. This common kernel is thus a normal subgroup of both K' and $\underline{H}$. Recall that a normal subgroup of a product of compact simple Lie groups can be written as a product of simple factors and finite central groups [BoreldS49, p. 205]. Since $K'/\ker\alpha|_{K'} = K_{\mathrm{eff}}$ is simple, it follows this kernel contains all but one simple factor of K', which we call $\widetilde{K}_{\mathrm{eff}}$, and the composite $\widetilde{K}_{\mathrm{eff}} \hookrightarrow K' \to \alpha(K') = K_{\mathrm{eff}}$ is a covering of $\mathrm{SO}(2n+1)$ or G_2. In the latter case one can only have $\widetilde{K}_{\mathrm{eff}} \cong G_2$ again and in the former one can have $\widetilde{K}_{\mathrm{eff}} \cong \mathrm{SO}(2n+1)$ or $\mathrm{Spin}(2n+1)$. Write L for the identity component of $\ker\alpha|_{K'}$, so that the latter is the product $L \times Z(\widetilde{K}_{\mathrm{eff}})$.

Now as $L \leqslant \underline{H}$ contains all but one direct factor of K' and H is a closed subgroup of K', it follows $\underline{H}$ is the direct product of L and $\widetilde{H}_{\mathrm{eff}} = \widetilde{K}_{\mathrm{eff}} \cap \underline{H}$. Since $\widetilde{H}_{\mathrm{eff}}/\widetilde{K}_{\mathrm{eff}} \to K'/\underline{H} \to K_{\mathrm{eff}}/H_{\mathrm{eff}}$ is a diffeomorphism, $\ker(\widetilde{K}_{\mathrm{eff}} \to K_{\mathrm{eff}})$ and $\ker(\widetilde{H}_{\mathrm{eff}} \to H_{\mathrm{eff}})$ have the same cardinality, giving the classification of possible pairs $(\widetilde{K}_{\mathrm{eff}}, \widetilde{H}_{\mathrm{eff}})$. □

Corollary 3.8. *In the situation and notation of Lemma 3.7 suppose additionally that if $\widetilde{K}_{\mathrm{eff}} \cong \mathrm{Spin}(2n+1)$, then the composition $F \hookrightarrow K \twoheadrightarrow \widetilde{K}_{\mathrm{eff}}$ is trivial. Then the inclusion $H \longhookrightarrow K$ is given up to isomorphism as*

$$\widetilde{H}_{\mathrm{eff}} \times (L \otimes_F A) \longhookrightarrow \widetilde{K}_{\mathrm{eff}} \times (L \otimes_F A).$$

Proof. We already have an equivalent expression $\underline{H} \otimes_F A \longhookrightarrow K' \otimes_F A$ by Lemma 3.6, and Lemma 3.7 lets us write $\underline{H} \longhookrightarrow K'$ as $\widetilde{H}_{\mathrm{eff}} \times L \longhookrightarrow \widetilde{K}_{\mathrm{eff}} \times L$, where the pair $(\widetilde{K}_{\mathrm{eff}}, \widetilde{H}_{\mathrm{eff}})$ is tightly prescribed. If $\widetilde{K}_{\mathrm{eff}}$ is $\mathrm{SO}(2n+1)$ or G_2, the only common central element of $\widetilde{H}_{\mathrm{eff}}$ and $\widetilde{K}_{\mathrm{eff}}$; otherwise we invoke the assumption $F \to \widetilde{K}_{\mathrm{eff}}$ is trivial; and either way we conclude $F = K' \cap Z(K)_0$ is contained entirely within L and we can pull out $\underline{H} \longhookrightarrow K'$. □

Proposition 3.9. *Suppose a compact, connected Lie group K and closed, connected subgroup H are given such that K/H is homeomorphic to an even-dimensional sphere, and suppose K' is the direct product of a simply-connected group and some number of factors $\mathrm{SO}(\mathrm{odd})$. If $\widetilde{K}_{\mathrm{eff}}$ denotes the unique direct factor of K' surjecting onto the image of the action map $K \longmapsto \mathrm{Homeo}\, K/H$, suppose additionally that the composition $K' \cap Z(K)_0 \hookrightarrow K \twoheadrightarrow \widetilde{K}_{\mathrm{eff}}$ is trivial (this condition is automatically satisfied unless $\widetilde{K}_{\mathrm{eff}} \cong \mathrm{Spin}(2n+1)$). Then splittings as in Proposition 3.3 can be chosen compatibly so that $RK \longrightarrow RH$ is identified with $R\underline{K} \otimes R(A/F) \longrightarrow R\underline{H} \otimes R(A/F)$.*

Proof. By Corollary 3.8, one can identify the restriction $RK \longrightarrow RH$ with $(R\widetilde{K}_{\mathrm{eff}} \to R\widetilde{H}_{\mathrm{eff}}) \otimes \mathrm{id}_{R(L \otimes_F A)}$. Thus in the proof of Proposition 3.3 one can take the splittings $R\underline{K} \longrightarrow RK$ and $R\underline{H} \longrightarrow RH$ to respectively be

$$R\widetilde{K}_{\mathrm{eff}} \otimes RL \xrightarrow{\mathrm{id} \otimes \varphi} R\widetilde{K}_{\mathrm{eff}} \otimes R(L \otimes_F A),$$

$$R\widetilde{H}_{\mathrm{eff}} \otimes RL \xrightarrow{\mathrm{id} \otimes \varphi} R\widetilde{H}_{\mathrm{eff}} \otimes R(L \otimes_F A)$$

for the same choice of $\varphi\colon RL \longrightarrow R(L \otimes_F A)$. This makes the evident square commute by definition, so $RK \longrightarrow RH$ can now be identified with $R\widetilde{K}_{\mathrm{eff}} \otimes RL \otimes R(A/F) \longrightarrow R\widetilde{H}_{\mathrm{eff}} \otimes RL \otimes R(A/F)$ as we wanted. □

3.2. Lemmas for odd spheres

The results we need for the case the homogeneous sphere K/H is odd-dimensional all follow from the splitting proposition 3.3 once we show $RK \longrightarrow RH$ is split surjective.

Proposition 3.10. *Let $H \leqslant K$ be connected, compact Lie groups such that $K/H \approx S^1$ and RK' is a polynomial ring. Then $RK \longrightarrow RH$ is a surjection and can be written*

$$RH[t^{\pm 1}] \xrightarrow{t \mapsto 1} RH,$$

where $t\colon K^{\mathrm{ab}} \twoheadrightarrow K^{\mathrm{ab}}/H^{\mathrm{ab}} \xrightarrow{\sim} \mathrm{U}(1)$ pulls back one of the generators of $R(K^{\mathrm{ab}}/H^{\mathrm{ab}})$ and is transcendental over RH.

Proof. Consider the diagram

$$\begin{array}{ccccccccc} 1 & \to & H' & \longrightarrow & H & \longrightarrow & H^{\mathrm{ab}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & K' & \longrightarrow & K & \longrightarrow & K^{\mathrm{ab}} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & K/H & \to & K/K'H, & & \end{array} \tag{3.11}$$

whose first two rows are exact sequences and whose first two rows and second column are fibrations. Since π_2 of a Lie group is zero, and $\pi_1(H')$ and $\pi_1(K')$ are finite, we see $\pi_1(K/H) \otimes \mathbb{Q} \to \pi_1(K/K'H) \otimes \mathbb{Q}$ is an isomorphism, so the torus $K^{\mathrm{ab}}/H^{\mathrm{ab}} = K/K'H$ is a circle. Particularly, it is one-dimensional, so counting other dimensions, we have $\dim K' = \dim H'$, meaning K'/H' is a connected 0-manifold and hence $K' = H'$.

The exact sequences of representation rings resulting from the first two rows of (3.11) split by Proposition 3.3. These splittings are not natural, but since $RK' \longrightarrow RH'$ is an isomorphism, we can choose the liftings compatibly so that the following diagram commutes:

$$\begin{array}{ccccccccc} & & & \nearrow & RK^{\mathrm{ab}} \otimes RK' & \nwarrow & & & \\ & & & & \downarrow \wr & & & & \\ \mathbb{Z} & \to & RK^{\mathrm{ab}} & \longrightarrow & RK & \longrightarrow & RK' & \to & \mathbb{Z} \\ & & \downarrow & & \downarrow & & \downarrow \wr & & \\ \mathbb{Z} & \to & RH^{\mathrm{ab}} & \longrightarrow & RH & \longrightarrow & RH' & \to & \mathbb{Z}. \\ & & & & \uparrow \wr & & & & \\ & & & \searrow & RH^{\mathrm{ab}} \otimes RH' & \swarrow & & & \end{array} \tag{3.12}$$

Since $RK^{\mathrm{ab}} \longrightarrow RH^{\mathrm{ab}}$ is induced by the inclusion $H^{\mathrm{ab}} \hookrightarrow K^{\mathrm{ab}}$ of a codimension-1 subtorus and monomorphisms between tori admit retractions, we have $RK^{\mathrm{ab}} \cong RH^{\mathrm{ab}} \otimes R(K^{\mathrm{ab}}/H^{\mathrm{ab}}) \cong RH^{\mathrm{ab}}[t^{\pm 1}]$ and the result follows. □

Proposition 3.13. *Let $H \leqslant K$ be connected, compact Lie groups such that K/H is a sphere of odd dimension 3 or more and RK' is a polynomial ring. Then $RK \longrightarrow RH$ is a surjection and if the unique direct factor $\widetilde{K}_{\mathrm{eff}}$ of K' surjecting onto the image of the action map in* $\mathrm{Homeo}\, K/H$ *is not of the form* $\mathrm{SO}(\mathrm{odd})$, *then $RK \longrightarrow RH$ can be written as*

$$RH[\bar{\rho}] \xrightarrow{\bar{\rho} \mapsto 0} RH,$$

where $\bar{\rho}$ is transcendental over RH and equals $\rho - \dim \rho$ for a K-representation ρ, trivial on H, such that the induced continuous map $K/H \longrightarrow \mathrm{U}$ represents the fundamental class of K/H.

Proof. In (3.11) the bottom map now is a fibering of an odd sphere over a torus, which is only possible if the torus in question is zero-dimensional. Thus $H^{\mathrm{ab}} \longrightarrow K^{\mathrm{ab}}$ is a homeomorphism, so $H' = \ker(H \to H^{\mathrm{ab}})$ and $H \cap K' = \ker(H \to K \to K^{\mathrm{ab}})$ are equal. Since $K/K'H$ is trivial and the fiber of the trivial map $K/H \longrightarrow K/K'H$ is $K'H/H \cong K'/(K' \cap H) = K'/H'$, it follows $K'/H' \longrightarrow K/H$ is a homeomorphism. By the following Proposition 3.14, one has $RK' \longrightarrow RH'$ a surjection of the form $RH'[\bar\rho] \xrightarrow{\sim} RK' \to RH'$ if the group $\widetilde{K}'_{\mathrm{eff}}$ of that lemma can be taken simply-connected, so Proposition 3.5 applies with A the maximal central torus of H and $\underline{K} = K'$ and $\underline{H} = H'$.

To show the generator has claimed property, recall that the Hodgkin map $\beta\colon R\Gamma \longrightarrow K^*(\Gamma)$ is functorial, factors through the module of indecomposables $QR\Gamma$, and induces isomorphisms $\Lambda_{\mathbb{Z}}[QR\Gamma] \xrightarrow{\sim} K^*(\Gamma)$ if $\pi_1(\Gamma)$ is torsion-free, as we now assume $\pi_1(K)$ (and hence $\pi_1(H)$) is. Thus $i^*\colon K^*(K) \longrightarrow K^*(H)$ is a surjection. A result of Minami [Min75, Prop. 4.1] then says $K^*(K/H)$ is the exterior algebra on the homotopy class $\overline{\beta}(\rho)$ of the composition $K/H \to \mathrm{U}(V_\rho) \hookrightarrow \mathrm{U}$ for an element $\rho \in RK$ whose class in QRK generates $\ker Q(RK \to RH)$. □

We have separated out the harder part of the preceding proof into that of the following result.

Proposition 3.14. *Let $H \leqslant K$ be connected, compact Lie groups such that K/H is a sphere of odd dimension 3 or more and RK' is a polynomial ring. Then the map $RK' \longrightarrow RH'$ is an augmentation-preserving split surjection. If the unique direct factor $\widetilde{K}_{\mathrm{eff}}$ of K' surjecting onto the image of the action map in* Homeo K/H *is not of the form* SO(odd), *then $RK' \longrightarrow RH'$ can be written as $RH'[\bar\rho] \twoheadrightarrow RH'$ for a judicious choice of section $RH' \rightarrowtail RK'$ and algebraically independent generator $\bar\rho$.*

Proof. Recall K' is a direct product of simply-connected simple groups and odd special orthogonal groups [Ste75] and recall the groups $\widetilde{K}_{\mathrm{eff}}$ and $\widetilde{H}_{\mathrm{eff}}$ from that that proof. As there, we have $\underline{H} = \alpha^{-1}(H_{\mathrm{eff}}) \cap K' = H'$ and we may write $RK' \longrightarrow RH'$ as $\mathrm{id}_{RL} \otimes (R\widetilde{K}'_{\mathrm{eff}} \to R\widetilde{H}'_{\mathrm{eff}})$. We need only analyze the last factor. Augmentation-preservation is just the fact restriction of representations preserves dimension, so it remains only to see $R\widetilde{K}'_{\mathrm{eff}} \longrightarrow R\widetilde{H}'_{\mathrm{eff}}$ is a surjection of the claimed form. This comes down to a short case analysis, as the entire list of realizations of an odd-dimensional sphere as the orbit space of an effective action of a compact, connected Lie group is the following [Besse, Ex. 7.13][GrWZ08, Table C, p. 104], where the balanced product notation $\otimes_{\mathbb{Z}/2}$ is as explained in Definition 3.1:

$$\begin{aligned}
S^{4n-1} &= \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)} = \frac{\mathrm{U}(1) \underset{\mathbb{Z}/2}{\otimes} \mathrm{Sp}(n)}{\Delta\mathrm{U}(1) \underset{\mathbb{Z}/2}{\otimes} \mathrm{Sp}(n-1)} = \frac{\mathrm{Sp}(1) \underset{\mathbb{Z}/2}{\otimes} \mathrm{Sp}(n)}{\Delta\mathrm{Sp}(1) \underset{\mathbb{Z}/2}{\otimes} \mathrm{Sp}(n-1)},\\
S^{2n-1} &= \mathrm{U}(n)/\mathrm{U}(n-1) = \mathrm{SU}(n)/\mathrm{SU}(n-1) = \mathrm{SO}(2n)/\mathrm{SO}(2n-1),\\
S^{15} &= \mathrm{Spin}(9)/\mathrm{Spin}(7),\\
S^{7} &= \mathrm{Spin}(7)/G_2.
\end{aligned} \tag{3.15}$$

Our task is made easier by the λ-ring structure on $R(-)$ induced by exterior powers: because the rings in question are largely generated by exterior powers of the standard representation σ, much of the work is done when we find σ in the image.

- For $R\mathrm{Sp}(n) \longrightarrow R\mathrm{Sp}(n-1)$ we have $\sigma \longmapsto \sigma + 2$ and for $R\mathrm{SU}(n) \longrightarrow R\mathrm{SU}(n-1)$ we have $\sigma \longmapsto \sigma + 1$. Now σ generates $R\mathrm{Sp}(n)$ and $R\mathrm{SU}(n)$ as λ-rings, so we already see the map is surjective.

In fact, the images of $\sigma, \ldots, \lambda^{n-1}\sigma$ generate the codomain in either case, since $\lambda^j(\sigma+2) = \lambda^j\sigma + 2\lambda^{j-1}\sigma + 1$ for $j \geqslant 2$ and $\lambda^j(\sigma+1) = \lambda^j\sigma + \lambda^{j-1}\sigma$ for $j \geqslant 1$.[11] It follows the image of $\lambda^n\sigma$ is also the image of some polynomial p in the lower $\lambda^j\sigma$, so we may rewrite the domain as $\mathbb{Z}[\sigma, \ldots, \lambda^{n-1}\sigma][\lambda^n\sigma - p]$ to obtain an expression of the claimed form.

- Writing $R\mathrm{Spin}(2n) \longrightarrow R\mathrm{Spin}(2n-1)$ as $\mathbb{Z}[\sigma, \ldots, \lambda^{n-2}\sigma, \Delta_-, \Delta_+] \longrightarrow \mathbb{Z}[\sigma, \ldots, \lambda^{n-2}\sigma, \Delta]$, where σ is the composition of the double cover with the defining representation of the special orthogonal group, $\Delta_\pm$ are the half-spin representations, and Δ is the spin representation, we have $\sigma \longmapsto \sigma + 1$ and $\Delta_\pm \longmapsto \Delta$ [BrötD, Prop. VI.6.1].

  By the same argument as before, the map is a bijection when restricted to $\mathbb{Z}[\sigma, \ldots, \lambda^{n-1}\sigma, \Delta_-]$, and we may replace the last generator by $\Delta_+ - \Delta_-$ to obtain the desired expression.

- The restriction $R\mathrm{SO}(2n) \longrightarrow R\mathrm{SO}(2n-1)$ is surjective because representations of $\mathrm{SO}(2n-1)$ descend from representations of $\mathrm{Spin}(2n-1)$ such that $-1 \in \mathrm{Spin}(2n-1)$ acts trivially, and we have just seen the map $R\mathrm{Spin}(2n) \longrightarrow R\mathrm{Spin}(2n-1)$ is surjective.[12]

  To get more specific expressions, we [BrötD, Prop. VI.6.6] may write the map as

$$\mathbb{Z}[\sigma, \ldots, \lambda^{n-1}\sigma, \lambda^n_+, \lambda^n_-]/(Q) \longrightarrow \mathbb{Z}[\sigma, \ldots, \lambda^{n-1}\sigma], \tag{3.16}$$

  where $\lambda^n_\pm$ are the ± 1-eigenspaces of the Hodge star on $\lambda^n\sigma$ and

$$Q = (\overbrace{\lambda^n_+ + \lambda^{n-2}\sigma + \cdots}^{x})(\overbrace{\lambda^n_- + \lambda^{n-2}\sigma + \cdots}^{y}) - (\overbrace{\lambda^{n-1}\sigma + \lambda^{n-3}\sigma + \cdots}^{z})^2.$$

  We have a decomposition $\lambda^n\sigma = \lambda^n_+ + \lambda^n_-$ into irreducibles, and $\lambda^n\sigma \longmapsto \lambda^n\sigma + \lambda^{n-1}\sigma = (\lambda^{n-1}\sigma)^\vee + \lambda^{n-1}\sigma = 2\lambda^{n-1}\sigma$ in $R\mathrm{SO}(2n-1)$ since the fundamental representations of $\mathrm{SO}(2n-1)$ are self-dual, so it follows that both of $\lambda^n_\pm$ are sent to $\lambda^{n-1}\sigma$. If we rewrite $R\mathrm{SO}(2n)$ as $\mathbb{Z}[\sigma, \ldots, \lambda^{n-2}\sigma][x, y, z]/(xy - z^2)$, we see each of x, y, z maps to $w = \sum_{j \leqslant n-1} \lambda^j\sigma$ (so that in particular Q maps to 0), and the map can be described as

$$\mathbb{Z}[\sigma, \ldots, \lambda^{n-2}\sigma][x, y, z]\big/(xy - z^2) \longrightarrow \mathbb{Z}\big[\sigma + 1, \ldots, \lambda^{n-2}(\sigma + 1)\big][w]. \tag{3.17}$$

  As the restriction $\mathbb{Z}[\sigma, \ldots, \lambda^{n-2}\sigma] \longrightarrow \mathbb{Z}\big[\sigma + 1, \ldots, \lambda^{n-2}(\sigma+1)\big]$ is an isomorphism, the map as a whole is split by additionally sending $w \longmapsto z$.

- One [vanL] can write $R\mathrm{Spin}(7) \longrightarrow RG_2$ as

$$\begin{aligned} \mathbb{Z}[\sigma, \lambda^2\sigma, \delta] &\longrightarrow \mathbb{Z}[\sigma, \mathrm{Ad}], \\ \sigma &\longmapsto \sigma, \\ \delta &\longmapsto 1 + \sigma, \\ \mathrm{Ad} = \lambda^2\sigma &\longmapsto \lambda^2\sigma = \sigma + \mathrm{Ad}\,. \end{aligned}$$

  Particularly, one can obtain the desired expression by exchanging the generator $\lambda^2\sigma$ for $\lambda^2\sigma - \sigma$ and δ for $\delta - \sigma - 1$.

[11] In general $\lambda^n(x+y) = \sum_{i+j=n} \lambda^i x \cdot \lambda^j y$, and for $m \in \mathbb{N}$ one has $\lambda^j m = \binom{m}{j}$.

[12] We will not use this case further, as $\mathrm{SO}(2n)$ is not simply-connected, but it is worth laying out clearly.

- One [VZ09, vanL] can write $R\mathrm{Spin}(9) \longrightarrow R\mathrm{Spin}(7)$ as

$$\begin{aligned}\mathbb{Z}[\sigma, \lambda^2\sigma, \lambda^3\sigma, \Delta] &\longrightarrow \mathbb{Z}[\sigma, \lambda^2\sigma, \delta],\\ \sigma &\longmapsto \delta + 1,\\ \Delta &\longmapsto \delta + \sigma + 1.\end{aligned}$$

  Then we have $\lambda^2(\sigma - 1) \longmapsto \lambda^2\delta = \sigma + \lambda^2\sigma$ and $\lambda^3(\sigma - 1) \longmapsto \sigma\delta - \delta$. Thus we can take instead as generators

$$\begin{aligned}\sigma - 1 &\longmapsto \delta,\\ \Delta - \sigma &\longmapsto \sigma,\\ \lambda^2(\sigma - 1) - (\Delta - \sigma) &\longmapsto \lambda^2\sigma,\\ \lambda^3(\sigma - 1) - (\Delta - \sigma - 1)(\sigma - 1) &\longmapsto 0.\end{aligned}$$

□

Remark 3.18. The two "exceptional" homogeneous spheres can be understood as follows. Recall that the compact exceptional group G_2 can be seen as the group of $\mathbb{R}$-algebra automorphisms of the octonions $\mathbb{O}$. The map $G_2 \rightarrowtail \mathrm{Spin}(7)$ lifts the inclusion $G_2 \hookrightarrow \mathrm{SO}(7)$ arising from restriction of the defining action to the subspace of pure imaginaries. For the map $\mathrm{Spin}(7) \rightarrowtail \mathrm{Spin}(9)$ yielding S^{15}, since $\pi_1\mathrm{Spin}(7) = 1$, one lifts the spin representation $\delta\colon \mathrm{Spin}(7) \rightarrowtail \mathrm{SO}(8)$ to $\mathrm{Spin}(7) \rightarrowtail \mathrm{Spin}(8)$, then follows with the map $\mathrm{Spin}(8) \hookrightarrow \mathrm{Spin}(9)$ double-covering the block-diagonal inclusion $\mathrm{SO}(8) \oplus [1] \hookrightarrow \mathrm{SO}(9)$.

The author learned these explanations from Jason DeVito.

Remark 3.19. The proof of Proposition 3.14 was originally routed through the following statement:

For any surjection $\varphi\colon A \twoheadrightarrow B$ of polynomial rings respectively in $m \geqslant n$ indeterminates over a commutative base ring k, one can choose an algebraically independent set $x_1, \ldots, x_n, y_{n+1}, \ldots, y_m$ of polynomial generators for A over k such that φ sends $y_j \longmapsto 0$ and restricts to an isomorphism $k[x_1, \ldots, x_n] \xrightarrow{\sim} B$.

This innocuous-sounding claim is true for graded maps of graded rings over $k = \mathbb{Q}$ and open for ungraded maps over $k = \mathbb{C}$. In algebro-geometric language, the special case $m = n + 1$ we use in this paper is the *Abhyankar–Sathaye embedding conjecture* [AbM75, Sat76, RusSat13, Pop15, Wendt], which states that any embedding $\mathbb{A}^n_{\mathbb{C}} \rightarrowtail \mathbb{A}^{n+1}_{\mathbb{C}}$ is taken to the standard embedding by some automorphism of $\mathbb{A}^{n+1}_{\mathbb{C}}$. This is known at present for $n = 1$ and several other special cases, and is closely related to the determination of the algebraic automorphism group $\mathrm{Aut}\,\mathbb{A}^m_{\mathbb{C}}$, which is still incomplete for $m \geqslant 3$.

3.3. Lemmas for even spheres

In case the homogeneous sphere K/H is even-dimensional, the restriction $RK \longrightarrow RH$ makes the RH a free module of rank two over RK.

Proposition 3.20. *Let $H \leqslant K$ be connected, compact Lie groups of equal rank such that K/H is an even-dimensional sphere and the semisimple component K' is the direct product of a simply-connected group and* $\mathrm{SO}(\mathrm{odd})$ *factors. Then RH is a free RK-module of rank two.*

Proof. Steinberg [Ste75], strengthening an earlier result of Pittie, shows that with our hypotheses, RH is free of rank $|W_K|/|W_H|$ over RK (he also provides a basis). To see the rank is two, note that by completion [CF18, Thm. 5.3], this is also the rank of $H^*(BH;\mathbb{Q})$ over $H^*(BK;\mathbb{Q})$, which is 2 by the collapse of the Serre spectral sequence of $K/H \to BH \to BK$ with rational coefficients. □

We will apply Lemma 3.6 in conjunction with a refinement due to Adem and Gómez of the Steinberg basis theorem.

Theorem 3.21 (Adem–Gómez [AdG12, Thm. 3.5])**.** *Let G be a compact, connected Lie group with free abelian fundamental group and fix a choice Φ^+ of positive roots of G with respect to some maximal torus. Let $\mathscr{W} = (W_j)$ be a family of subgroups of $W = WG$, including the trivial group 1 and W itself, each generated by reflections in some subsystem Φ_j^+ of Φ^+, and for each j, write $\widetilde{W/W_j}$ for the set $\{w \in W : w\Phi_j^+ \subseteq \Phi^+\}$ of coset representatives for W/W_j. Suppose any pair W_j and W_k in $\mathscr{W}$ lie in a common supergroup $W_\ell \in \mathscr{W}$ such that $\widetilde{W/W_\ell} = \widetilde{W/W_j} \cap \widetilde{W/W_k}$.*

Then RT and the other $(RT)^{W_j}$ are all free modules over the subring $(RT)^W$ and respective bases $B \subseteq RT$ and $B_j \subseteq B$ respectively k of RT and the other $(RT)^{W_j}$ as free $(RT)^W$-submodules of RT can be chosen such that $B_k \subseteq B_j$ whenever $W_k \geqslant W_j$ in $\mathscr{W}$ and the inclusion $(RT)^{W_k} \longleftrightarrow (RT)^{W_j}$ is the map of free $(RT)^W$-submodules of RT induced by the inclusion $B_k \longleftrightarrow B_j$.

To apply this theorem we require some facts about extensions of root systems.

Lemma 3.22. *A lattice of Killing–Cartan type A_2 extends to a G_2 lattice in a unique way.*

Proof. If view the A_2 lattice as the vectors $(a_1, a_2, a_3) \in \mathbb{Z}^3$ with $a_1 + a_2 + a_3 = 0$, a new simple root α in an extending G_2 lattice must have length $\sqrt{6}$ and inner products with A_2 lattice elements divisible by 3. We would not rob the reader of the simple joy of verifying only $\pm(2, -1, -1)$, $\pm(-1, 2, -1)$, and $\pm(-1, -1, 2)$ do the job. □

Lemma 3.23. *A lattice of Killing–Cartan type D_n extends to a B_n lattice in*

$$\begin{cases} \textit{a unique way} & \textit{if } n \neq 4, \\ \textit{precisely two ways} & \textit{if } n = 4. \end{cases}$$

Proof. The standard D_n lattice in $\mathbb{R}^n$ is spanned by roots $e_j \pm e_k$, and so is given by those integer linear combinations $\sum a_j e_j$ of the standard basis vectors $e_j \in \mathbb{R}^n$ for which $\sum a_j$ is even. A new root α in an extending B_n lattice must have length 1 and inner product with all such vectors an integer, but the only vectors satisfying this are generally $\pm e_j$ and additionally for B_4 the vectors $\sum_{j=1}^4 \pm \frac{1}{2} e_j$. The standard B_n comes from adding a simple root of the first form to a D_n root system, while it is easy to check the rows of the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

are also simple roots for a B_4 root system. □

The union of these two lattices contains an F_4 root system

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

and so generates an F_4 lattice. Indeed, there are two distinct $\mathrm{Spin}(9)$ subgroups $K^{\pm}$ of the group $G = F_4$ meeting in a $\mathrm{Spin}(8) = H$ and witnessing this root data [GrWZ08, Table E, p. 125]. The resulting double mapping cylinder is S^{25}.

Lemma 3.24. *The family of Weyl groups $(WG, WK^-, WK^+, WH, 1)$ corresponding to the cohomogeneity-one action in the preceding paragraph meets the hypotheses of Theorem 3.21.*

Proof. We note that F_4 is simply-connected. The coset condition of Theorem 3.21 is satified automatically if, in that notation, one of W_j and W_k contains the other, so we only need to check that for $W_j = WK^-$ and $W_k = WK^+$, we can take $W_\ell = WG$. But, as is easy to ask a computer to check [Car], if one chooses the positive roots Φ^+F_4 of the F_4 root system to be e_j, $e_j \pm e_k$, and $\frac{1}{2}(1, \pm 1, \pm 1, \pm 1)$ and the positive roots $\Phi^+K^{\pm}$ of the smaller groups to be subsets of these, then the sets $\{w \in WF_4 : w\Phi^+K^{\pm} \subsetneq \Phi^+F_4\}$ of coset representatives of $WF_4/WK^{\pm}$ meet only in the neutral element. □

We will need to apply Theorem 3.21 to one other case, the system of subgroups of $\mathrm{Sp}(3)$ given by the block-diagonal subgroups $K^- = \mathrm{Sp}(2) \oplus \mathrm{Sp}(1)$ and $K^+ = \mathrm{Sp}(1) \oplus \mathrm{Sp}(2)$, which meet in the diagonal $H = \mathrm{Sp}(1)^{\oplus 3}$. All share as a maximal torus $T = \mathrm{U}(1)^{\oplus 3}$. It is easy to see that the roots of the larger groups in T generate an C_3 lattice, and under the standard identification of $W\mathrm{Sp}(3)$ with $\Sigma_3 \ltimes \{\pm 1\}^3 < \mathrm{Aut}\,\mathbb{R}^3$, the subgroups WK^- and WK^+ become respectively $\langle (1\,2) \rangle \cdot \{\pm 1\}^3$ and $\langle (2\,3) \rangle \cdot \{\pm 1\}^3$, while WT is simply $\{\pm 1\}^3$.

Lemma 3.25. *The family of Weyl groups $(WG, WK^-, WK^+, WH, 1)$ corresponding to the cohomogeneity-one action in the preceding paragraph meets the hypotheses of Theorem 3.21.*

Proof. Note that $\mathrm{Sp}(3)$ is simply-connected. As before, the only pair of containment-incomparable subgroups under consideration is $\{WK^-, WK^+\}$, and one checks [Car] the sets of coset representatives $\{w \in WC_3 : w\Phi^+K^{\pm} \subsetneq \Phi^+C_3\}$ for $WC_3/WK^{\pm}$ meet only in 1. □

4. The case when one sphere is odd-dimensional

We now put the algebra of the previous section to use to obtain specializations of Theorem 2.11. In this section, at least one of the homogeneous spheres $K^{\pm}/H$ is odd-dimensional.

Theorem 4.1. *Let M be the double mapping cylinder of the span $G/H \rightrightarrows G/K^{\pm}$ for inclusions $H \rightrightarrows K^{\pm} \rightrightarrows G$ of closed, connected subgroups of a compact Lie group G such that $K^{\pm}/H$ are spheres and the fundamental groups $\pi_1(K^{\pm})$ are free abelian.*

(a) Assume that both K^+/H and K^-/H are odd-dimensional. Then we have an RG-algebra isomorphism of $K^*_G(M) = K^0_G(M)$ with one of

$$\frac{RH[t_-^{\pm 1}, t_+^{\pm 1}]}{(t_- - 1)(t_+ - 1)}, \qquad \frac{RH[t_-^{\pm 1}, \overline{\rho}_+]}{(t_- - 1)(\overline{\rho}_+)}, \qquad \frac{RH[\overline{\rho}_-, t_+^{\pm 1}]}{(\overline{\rho}_-)(t_+ - 1)}, \qquad \frac{RH[\overline{\rho}_-, \overline{\rho}_+]}{(\overline{\rho}_- \overline{\rho}_+)},$$

where we identify $RK^{\pm}$ with the Laurent polynomial ring $RH[t_{\pm}^{\pm 1}]$ when $\dim K^{\pm}/H = 1$ and with the polynomial ring $RH[\overline{\rho}_{\pm}]$ when $\dim K^{\pm}/H \geqslant 3$.

(b) Assume K^+/H is odd-dimensional and K^-/H is even-dimensional. Then we have an RG-algebra isomorphism of $K^*_G(M) = K^0_G(M)$ with

$$RK^- \oplus (t-1)RH[t^{\pm 1}] < RH[t^{\pm 1}] \cong RK^+ \qquad \text{or} \qquad RK^- \oplus \overline{\rho} RH[\overline{\rho}] < RH[\overline{\rho}] \cong RK^+,$$

where we identify RK^+ with $RH[t_{\pm}^{\pm 1}]$ if $\dim K^+/H = 1$ and with $RH[\overline{\rho}_{\pm}]$ if $\dim K^+/H \geqslant 3$. The product in either case is determined by the restriction $RK^- \rightarrowtail RH$.

In all cases the RG-module structure is determined by restriction.

Remark 4.2. In terms of representations, t is the class of the representation $K^+ \to (K^+)^{\mathrm{ab}}/H^{\mathrm{ab}} \xrightarrow{\sim} \mathrm{U}(1)$, and similarly for $t_{\pm}$. Likewise, $\overline{\rho}$ is the reduction $\rho - \dim\rho$ of a complex K^+-representation $\rho\colon K^+ \longrightarrow \mathrm{U}(V_\rho)$, trivial when restricted to H, such that the class $\overline{\beta}(\rho)$ represented by the composition $K^+/H \to \mathrm{U}(V_\rho) \hookrightarrow \mathrm{U}$ generates $K^1(K^+/H)$, and similarly for $\overline{\rho}_{\pm}$.

Proof of Theorem 4.1. We use the description of $K^*_G(M)$ given in Theorem 2.11. In both cases, $K^1_G(M) = 0$ since $RK^+ \longrightarrow RH$ is surjective, so $K^*_G(M) = K^0_G(M) \cong RK^- \underset{RH}{\times} RK^+$.

(a) Recall from Theorem 0.4 that $RK^- \longrightarrow RH$ is an injection and from Propositions 3.10 and 3.13 that the map $RK^+ \xrightarrow{\sim} RH[\bar\rho] \to RH$ or $RK^+ \xrightarrow{\sim} RH[t^{\pm 1}] \to RH$ is reduction modulo $(\bar\rho)$ or $(t-1)$. We prove the latter case; the former is similar. Then the fiber product is the subring of $RH[t^{\pm 1}] \times RK^+$ consisting of the direct summands $\big\{(\sigma,\sigma) \in RK^+ \times RK^+\big\}$ and $(t-1)RH[t^{\pm 1}] \times \{0\}$. We may identify the former with $RK^+ < RH < RH[t^{\pm 1}]$ and the latter with $(t-1)RH[t^{\pm 1}] \lhd RH[t^{\pm 1}]$ and the two interact multiplicatively via the rule

$$\sigma \cdot (t-1)f \longleftrightarrow (\sigma,\sigma)\cdot\big((t-1)f, 0\big) = \big((t-1)\sigma f, 0\big) \longleftrightarrow (t-1)\sigma f.$$

(b) We use Theorem 0.4 to make identifications $RK^- \cong RH[t^{\pm 1}]$ and $RK^+ \cong RH[\bar\rho]$ such that $RK^- \longrightarrow RH$ is reduction modulo $\bar t = t - 1$ and $RK^+ \longrightarrow RH$ modulo $\bar\rho$; the other cases are the same, *mutatis mutandis*. The fiber product can be identified as the subring of $RH[t^{\pm 1}] \times RH[\bar\rho]$ comprising the three direct summands

$$\big\{(\sigma,\sigma) \in RH \times RH\big\}, \qquad \bar t RH[t^{\pm 1}] \times \{0\}, \qquad \{0\} \times \bar\rho RH[\bar\rho].$$

Multiplication across summands is determined by the three rules

$$(\sigma,\sigma)\cdot(\bar t f^-, 0) = (\bar t f^- \sigma, 0), \qquad (\sigma,\sigma)\cdot(0, \bar\rho f^+) = (0, \bar\rho f^+ \sigma), \qquad (\bar t f^-, 0)\cdot(0,\bar\rho f^+) = (0,0),$$

so the map to $RH[t^{\pm 1}, \bar\rho]/(\bar t\bar\rho)$ sending $(\sigma + \bar t f^-, \sigma + \bar\rho f^+)$ to the class $\sigma + \bar t f^- + \bar\rho f^+ \pmod{\bar t\bar\rho}$ is a ring isomorphism. □

Remark 4.3. This statement is obviously not the most one can say, in that it can be extended using the extraneous description (3.17) of $R\mathrm{SO}(2n) \longrightarrow R\mathrm{SO}(2n-1)$ in the proof of Proposition 3.14 to cover the cases where the image of one or more of $K^{\pm} \longrightarrow \operatorname{Homeo} K^{\pm}/H$ comes from an $\mathrm{SO}(\text{even})$ subgroup of $K^{\pm}$—but this is left as an exercise for the interested reader, if any, the current statement being long enough as it is.

Example 4.4. Let M be the double mapping cylinder associated to a diagram with $H = \mathrm{Spin}(7)$ included in $K^- = \mathrm{Spin}(8)$ via the standard inclusion and in $K^+ = \mathrm{Spin}(9)$ via the nonstandard embedding with $K^+/H = S^{15}$; the larger group G can be anything large enough, say F_4 or $\mathrm{Spin}(8) \times \mathrm{Spin}(9) = K^- \times K^+$. Then we have an explicit presentation

$$K^*_G(M) \cong \mathbb{Z}[\sigma, \lambda^2\sigma, \Delta, \bar\rho_-, \bar\rho_+]/(\bar\rho_-\bar\rho_+),$$

where in $R\mathrm{Spin}(8) \times R\mathrm{Spin}(9)$, the generators are represented by

$$\begin{aligned} \sigma &\longleftrightarrow (\sigma - 1, \Delta - \sigma), \\ \lambda^2\sigma &\longleftrightarrow (\lambda^2\sigma - \sigma - 1, \lambda^2(\sigma - 1) + \sigma - \Delta), \\ \Delta &\longleftrightarrow (\Delta_-, \sigma - 1), \\ \bar\rho_- &\longleftrightarrow (\Delta_+ - \Delta_-, 0), \\ \bar\rho_+ &\longleftrightarrow \big(0, \lambda^3(\sigma - 1) - (\Delta - \sigma - 1)(\sigma - 1)\big). \end{aligned}$$

in the manner described in Remark 2.14.

5. The case when both spheres are even-dimensional

In this section we obtain the specialization of Theorem 2.11 where both the homogeneous spheres $K^\pm/H$ are even-dimensional. ***The groups $K^\pm$ and H are assumed to satisfy this condition everywhere in this section.*** Particularly, K^-, K^+, and H all have the same rank. We will not have to assume that $\pi_1(K^\pm)$ is free abelian, but only that the commutator subgroup K' is the direct product of a simply-connected factor and a number of $\mathrm{SO}(\text{odd})$ factors. This is equivalent to assuming RK' is a polynomial ring [Ste75].

Notation 5.1. Occasionally we will write T for a maximal torus of some connected, compact Lie group Γ and use the fact that $R\Gamma \cong (RT)^{W\Gamma}$ by restriction [AtH61, §4.4], where $W\Gamma$ is the Weyl group of Γ. Particularly, when $K^\pm/H$ are even-dimensional spheres, $RH = (RT)^{WH}$ is of rank two over $RK^\pm = (RT)^{WK^\pm}$, so WH is an index-two subgroup of each of $WK^\pm$.

We start with two similar reduction lemmas which will save us time later.

Lemma 5.2. *Suppose $K^\pm, H$ are compact and connected and there are groups $\underline{K}^\pm \leqslant K^\pm$ and $L, \underline{H} \leqslant H$ such that $K^\pm = \underline{K}^\pm \times L$ and $H = \underline{H} \times L$ (we then write for short $(K^\pm, H) = (\underline{K}^\pm, \underline{H}) \times L$), and write $\underline{M}$ for the double mapping cylinder of $G/\underline{H} \rightrightarrows G/\underline{K}^\pm$. Then $K^*_G(M) \cong K^*_G(\underline{M}) \otimes RL$.*

Proof. This follows from Theorem 2.11 since the map $RK^- \times RK^+ \longrightarrow RH$ then factors as $(R\underline{K}^- \times R\underline{K}^+ \to R\underline{H}) \otimes \mathrm{id}_{RL}$. □

Lemma 5.3. *Suppose $K^\pm, H$ are compact and connected and there are groups $\underline{K}^\pm \leqslant K^\pm$ and $A, \underline{H} \leqslant H$ such that A is a torus central in both of $K^\pm$ and such that $K^\pm$ can be written as $\underline{K}^\pm \otimes_F A$ for the same finite subgroup $F \leqslant A$ and some closed subgroups $\underline{K}^\pm \leqslant K^\pm$. Then $H = \underline{H} \otimes_F A$. Suppose the pairs (K^+, H) and (K^-, H) both satisfy the conditions of Corollary 3.8. Writing $\underline{M}$ for the double mapping cylinder of $G/\underline{H} \rightrightarrows G/\underline{K}^\pm$, we then have $K^*_G(M) \cong K^*_G(\underline{M}) \otimes R(A/F)$.*

Proof. The first clause applies from Lemma 3.6 applied to both pairs $(K^\pm, H)$. The rest follows from Theorem 2.11 and Proposition 3.9 since the map $RK^- \times RK^+ \longrightarrow RH$ then factors as $(R\underline{K}^- \times R\underline{K}^+ \to R\underline{H}) \otimes \mathrm{id}_{R(A/F)}$. □

After application of these lemmas, it will follow from a case analysis that most of the time we are in one of two special situations. The easier of these two situations is when $K^- = K^+$.

Proposition 5.4. *Assume there exists w in the identity component $N_G(H)_0$ such that $K^+ = wK^-w^{-1}$, that $K^-/H = S^{2n}$ is a sphere of positive even dimension and the left K^--action is orientation-preserving. Then*

$$K^*_G(M) \cong RK^- \otimes K^*(S^{2n+1}).$$

Proof. Note that in this case [GrWZ08, p. 44], M is G-diffeomorphic to the double mapping cylinder of $G/H \rightrightarrows G/K^-$, so we may as well assume $K^+ = K^-$. Then we may apply Theorem 2.11, noting that $RK^- \cap RK^+ = RK$ and that by Proposition 3.20,

$$\frac{RH}{RK^- + RK^+} = \frac{RK^-\{1,\rho\}}{RK^-} \cong RK^-\{\rho\}. \qquad \square$$

Remark 5.5. Forgetting the manifold itself and proceeding in terms of representation theory, we could also have noted that if $K > H$ share a maximal torus and w lies in $N_G(H)_0$, then wKw^{-1} also contains that torus, with respect to which $WK = W(wKw^{-1})$.

Proceeding more topologically, on the other hand, we could note that if $K^+ = K^- = K$, then the natural map $BH \longrightarrow BK$ allows us to define a sphere bundle $S(K/H) \to M_G \to BK$. The proof of the analogue for Borel cohomology [CGHM19, Prop. 5.2] worked by showing this bundle was cohomologically trivial, and it is to reflect this analogy that we retain the number n.

Remark 5.6. It is interesting to note that if we do not have $K^+ = K^-$, then $H = K^- \cap K^+$. To see this, first note that since $K^- \cap K^+$ and H share a maximal torus, $(K^- \cap K^+)/H$ is even-dimensional. But $(K^- \cap K^+)/H \to K^+/H \to K^+/(K^- \cap K^+)$ is a fibering of a sphere over a simplicial complex and by connected simplicial complexes, and Browder showed that when the fiber is none of S^1, S^3, or S^7, either the base or the fiber of such a bundle must be trivial [Brow63].

But this dichotomy does not lead to a dichotomy in expressions for $K^*_G(M)$. For example, the block-diagonal subgroup $H = \mathrm{SO}(4) \oplus [1]^{\oplus 2}$ of $G = \mathrm{SO}(6)$ is the intersection of $K^- = \mathrm{SO}(5) \oplus [1]$ and $K^+ = wK^-w^{-1}$ for $w = [1]^{\oplus 4} \oplus \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, which lies in $[1]^{\oplus 4} \oplus \mathrm{SO}(2) < N_G(H)_0$. Thus, up to diffeomorphism, the inclusion diagram (G, K^-, K^+, H) expresses the same double mapping cylinder M as the one instead taking $K^+ = K^- = \mathrm{SO}(5) \oplus [1]$.

The other easy-to-manage special case follows from a less trivial product decomposition.

Proposition 5.7. *Let connected, compact Lie groups $\underline{K}^\pm > \underline{H}^\pm$ be such that $\underline{K}^\pm/\underline{H}^\pm = S^{2n_\pm}$ are even-dimensional spheres. Write $H = \underline{H}^- \times \underline{H}^+$ and consider it in the natural way as a subgroup of $K^- = \underline{K}^- \times \underline{H}^+$, of $K^+ = \underline{H}^- \times \underline{K}^+$, and of $G = \underline{K}^- \times \underline{K}^+$. Then if M is the double mapping cylinder of $G/H \rightrightarrows G/K^\pm$, we have*

$$K^*_G M \cong RG \otimes \Lambda[z]$$

for a generator z of degree 1.

Proof. By Proposition 3.20, we know $R\underline{H}^\pm$ is free of rank two over $R\underline{K}^\pm$, say on bases $\{1, \sigma_\pm\}$. Then RK^-, RK^+, and RH are free over $RG = R\underline{K}^- \otimes R\underline{K}^+$ respectively on the bases

$$\{1 \otimes 1,\ \sigma_- \otimes 1\}, \qquad \{1 \otimes 1,\ 1 \otimes \sigma_+\}, \qquad \{1 \otimes 1,\ \sigma_- \otimes 1,\ 1 \otimes \sigma_+,\ \sigma_- \otimes \sigma_+\}.$$

Thus, by Theorem 2.11, we see $K^0_G(M)$ is the intersection of $RK^\pm|_H$, namely the free RG-module on $1 \otimes 1$, and $K^1_G(M) \cong RH/(RK^- + RK^+)$ is the free cyclic RG-module on $z = \delta(\sigma_- \otimes \sigma_+)$. Thus $K^*_G(M)$ is a free RG-module on $1 \in K^0_G(M)$ and $z \in K^1_G(M)$, and since $2z^2 = 0$ by antisymmetry and $K^*_G(M)$ is torsion-free, it follows $z^2 = 0$. $\square$

Remark 5.8. The manifold M is a sphere $S^{2n_-+2n_++1}$ under these conditions.[13] Indeed, the fiber over -1 is S^{2n_-}, that over 1 is S^{2n_+}, and in the interior the fiber is the product of the two, so M is the join $S^{2n_-} * S^{2n_+}$.

Example 5.9 ([Püt09, Sect. 4.3]). We use Proposition 5.4 to compute the equivariant cohomology of the space M arising from the inclusion diagram

$$(G, K^-, K^+, H) = \big(\mathrm{Sp}(2), \mathrm{Sp}(1)^2, \mathrm{Sp}(1)^2, \mathrm{Sp}(1) \times \mathrm{U}(1)\big).$$

Püttmann shows $H^*(M;\mathbb{Z}) \cong H^*(S^3;\mathbb{Z}) \otimes H^*(S^4;\mathbb{Z})$ using the Mayer–Vietoris sequence, so from the Atiyah–Hirzebruch spectral sequence we see $K^*(M) \cong K^*(S^3) \otimes K^*(S^4)$ as well. The restriction of the defining representation σ of $\mathrm{Sp}(1) < \mathbb{H}^\times$ on $\mathbb{H} \cong \mathbb{C} \oplus j\mathbb{C}$ to the maximal torus $\mathrm{U}(1) < \mathbb{C}^\times$ is $t + t^{-1}$, where t is the defining representation, so

$$K^1_G(M) \cong \frac{\mathbb{Z}[\sigma] \otimes \mathbb{Z}[t^{\pm 1}]}{\mathbb{Z}[\sigma] \otimes \mathbb{Z}[t + t^{-1}]} \cong \mathbb{Z}[\sigma] \otimes t\mathbb{Z}[t + t^{-1}] \cong R\big(\mathrm{Sp}(1)^2\big)[1]$$

as expected.

This action is equivariantly formal for Borel cohomology with integer coefficients [GoeM14, Cor. 1.3], and from Theorem 6.1, it is equivariantly formal for K^*_G too, but it is illuminating to show this explicitly by examining the forgetful map $K^*_G \longrightarrow K$ on the Mayer–Vietoris sequence of the standard cover. By the snake lemma, this amounts to checking the maps

$$R\Gamma \xrightarrow{\sim} K^0_G(G/\Gamma) \longrightarrow K^0(G/\Gamma)$$

taking a representation V_ρ of Γ to the bundle $G \otimes_\Gamma V_\rho \longrightarrow G/\Gamma$ are surjective for $\Gamma \in \{K^\pm, H\}$.[14] It is not hard to check this map takes $1 \otimes t \in R\big(\mathrm{Sp}(1) \times U(1)\big)$ to the tautological bundle γ over $\mathbb{C}\mathrm{P}^3$ and $1 \otimes \sigma \in R\big(\mathrm{Sp}(1)^2\big)$ to the tautological bundle ξ over $\mathbb{H}P^1$.[15] Since $H^*(\mathbb{C}\mathrm{P}^3) = \mathbb{Z}[c]/(c^3)$, where $c = c_1(\gamma)$, and c_1 induces an isomorphism $\widetilde{K}^0(\mathbb{C}\mathrm{P}^1) \xrightarrow{\sim} H^2(\mathbb{C}\mathrm{P}^1)$, this gives us surjectivity for H. As for $K^\pm$, since σ restricts to $\mathrm{U}(1)$ as $t + t^{-1}$, we see the pullback of ξ over $\mathbb{C}\mathrm{P}^3$ is $\gamma \oplus \gamma^\vee$. The total Chern class $1 + c_2(\tau) \in H^*(\mathbb{H}\mathrm{P}^1)$ hence pulls back to $(1+c)(1-c) \in H^*(\mathbb{C}\mathrm{P}^3)$. The Serre spectral sequence of $S^3/S^1 \to \mathbb{C}\mathrm{P}^3 \to \mathbb{H}\mathrm{P}^1$ collapses for degree reasons, so that $H^4(\mathbb{H}P^1) \to H^4(\mathbb{C}\mathrm{P}^3)$. Thus, since $-c^2$ generates $H^4(\mathbb{C}\mathrm{P}^3)$, also $c_2(\tau)$ generates $H^4(\mathbb{H}P^1)$. As

$$\widetilde{K}^0(S^4) \cong \widetilde{K}^4(S^4) \cong \widetilde{K}^0(S^0) = \mathbb{Z}$$

and the Chern character induces a natural isomorphism $K^* \otimes \mathbb{Q} \longrightarrow H^*(-;\mathbb{Q})$ on finite complexes, it follows $[\tau]$ generates $\widetilde{K}^0(S^4)$ as needed.

The desired simultaneous generalization of Propositions 5.4 and 5.7, specializing to Theorem 0.4 when $K^\pm$ are semisimple, is as follows.

Theorem 5.10. *Let M be the double mapping cylinder of the span $G/H \rightrightarrows G/K^\pm$ for inclusions $H \rightrightarrows K^\pm \rightrightarrows G$ of compact Lie groups such that $K^\pm$ are semisimple groups which are products of simply-connected groups and* $\mathrm{SO}(\text{odd})$ *factors and $K^\pm/H$ are even-dimensional spheres. Writing $\widetilde{K}^\pm_{\mathrm{eff}}$ for the*

[13] This will also hold if either sphere or both is odd-dimensional.

[14] In fact, applying the module structure in Theorem 2.11 to both sequences, it would be enough just to see $K^0_G M \longrightarrow K^0 M$ is surjective, and once we know $K^1(G/H) = K^1\mathbb{C}\mathrm{P}^3 = 0$, it would suffice to prove $RK \longrightarrow K^0(G/K)$ is surjective, but the same proof involves both maps.

[15] Note $\mathrm{Sp}(2) \longrightarrow S^7$ given by $A \longmapsto A \cdot \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ has stabilizer $\mathrm{Sp}(1) \oplus 1$ and transforms the action of $1 \oplus \mathrm{Sp}(1)$ to scalar right-multiplication on $S^7 \subsetneq \mathbb{H}^2$, so the total spaces of the bundles are $S^7 \otimes_{\mathrm{Sp}(1)} \mathbb{H}$ and $S^7 \otimes_{\mathrm{U}(1)} \mathbb{C}$.

unique direct factors of the commutator subgroups $(K^\pm)'$ *surjecting onto the images of the action maps* $K^\pm \longmapsto \operatorname{Homeo} K^\pm/H$*, suppose additionally that the compositions* $(K^\pm)' \cap Z(K^\pm)_0 \hookrightarrow K^\pm \twoheadrightarrow \widetilde{K}^\pm_{\mathrm{eff}}$ *are trivial (this is only relevant if one of* $K^\pm$ *is a* $\mathrm{Spin}(2n+1)$*). Then there exist an element* $z \in K^1_G(M)$ *and an RG-algebra isomorphism*

$$K^*_G(M) \;\cong\; (RK^-|_H \cap RK^+|_H) \otimes \Lambda[z],$$

where the injections $RK^\pm \longrightarrow RH$ *and the RG-module structure are given by restriction.*

The proof has been factored into as many Lie-theoretic lemmas and reduction steps as possible but still seems to unavoidably be a bit of a slog.

Proof of Theorem 5.10. Recall from the proof of Lemma 3.7 that the images $K^\pm_{\mathrm{eff}}$ of the action maps $\alpha^\pm \colon K^\pm \longrightarrow \operatorname{Homeo} K^\pm/H$ are by definition effective and hence must be $\mathrm{SO}(2n+1)$ or G_2, with the image of H being $\mathrm{SO}(2n)$ or $\mathrm{SU}(3)$ respectively [Besse, Ex. 7.13][GrWZ08, Table C, p. 104]. The effective images $H^\pm_{\mathrm{eff}} = \alpha^\pm(H)$ of H, in particular, determine $K^\pm_{\mathrm{eff}}$ uniquely up to isomorphism.

Most of the proof involves analyzing the configurations of these preimages after stripping away extra tensor factors to eventually arrive at a base case. The recurrent phrase "***factor out*** Π" means to apply Lemma 5.2 and analyze the remaining system of isotropy groups $\underline{K}^- \leftarrow \underline{H} \rightarrow \underline{K}^+$, whereas "***factor out*** A/F" means to apply Lemma 5.3. We say we have reduced to a ***join configuration*** if Proposition 5.7 applies, in which case that branch of the case analysis terminates, and similarly say we have reduced to a ***sphere bundle configuration*** if Proposition 5.4 applies. Beyond these base case schemata, there are a few exceptional base cases enumerated in Section 3.3, which as we have mentioned, all turn up as examples in the literature, and the case with $H^-_{\mathrm{eff}} \cong \mathrm{SO}(2) \cong H^+_{\mathrm{eff}}$.

0. *The case neither of* $H^\pm_{\mathrm{eff}}$ *is a circle*

As $K^\pm/H$ are even-dimensional spheres of dimension > 2, the long exact fibration sequence of $H \to K^\pm \to K^\pm/H$ induces isomorphisms $\pi_1 H \xrightarrow{\sim} \pi_1 K^\pm$. It follows that the inclusion of $A = Z(H)_0$ in H induces surjections $\pi_1 A \longrightarrow \pi_1 K^\pm$ and we can write $K^\pm$ as $(K^\pm)' \otimes_{F^\pm} A$ for $F^\pm = \ker\big((K^\pm)' \times A \twoheadrightarrow K^\pm\big)$. Since $K^\pm/H$ are spheres, by two applications of Lemma 3.6 we have $H' \otimes_{F^-} A = H = H' \otimes_{F^+} A$, so $F = F^- = F^+$. Thus the inclusions $H \rightrightarrows K^\pm$ factor as virtual product maps of the form $i_\pm \otimes_F \mathrm{id}_A$. Factoring out A/F, we need only analyze $K^*_G(M')$ for M' the double mapping cylinder of $G/(K^\pm)' \rightrightarrows G/H'$. We may thus adopt the notational convenience of assuming the groups $K^\pm$ of the original triple $(K^\pm, H)$ were semisimple.

Recall from the proof of Lemma 3.7 that the kernels of $\alpha^\pm|_H$ contain all but one simple factor of H, or all but two in case $H^\pm_{\mathrm{eff}} = \mathrm{SO}(4) \cong \mathrm{SO}(3)^2/\{\pm(I,I)\}$ is not simple. Thus we have product decompositions

$$K^\pm \cong \widetilde{K}^\pm_{\mathrm{eff}} \times \Pi^\pm,$$
$$H \cong \widetilde{H}^\pm_{\mathrm{eff}} \times \Pi^\pm,$$

where the ineffective kernels $\Pi^\pm := \ker \alpha^\pm$ are products of simply-connected and SO(odd) factors, their normal virtual complements $\widetilde{K}^\pm_{\mathrm{eff}} \leqslant K^\pm$ induce isomorphisms or double-coverings $\widetilde{K}^\pm_{\mathrm{eff}} \hookrightarrow K^\pm \twoheadrightarrow K^\pm_{\mathrm{eff}}$, and $\widetilde{H}^\pm_{\mathrm{eff}}$ are the intersections of H and $\widetilde{K}^\pm_{\mathrm{eff}}$, accordingly singly or doubly covering $H^\pm_{\mathrm{eff}}$ under $\alpha^\pm$.

- *Suppose that* $\widetilde{H}^+_{\mathrm{eff}} = \widetilde{H}^-_{\mathrm{eff}} =: \widetilde{H}_{\mathrm{eff}}$.

  Then $\Pi^+ = \Pi^-$ and we may factor it out. What remains is the pair of inclusions $\widetilde{H}_{\mathrm{eff}} \rightrightarrows \widetilde{K}^\pm_{\mathrm{eff}}$, so we examine the images of $R\widetilde{K}^\pm_{\mathrm{eff}} \rightrightarrows R\widetilde{H}_{\mathrm{eff}}$.

  - *Suppose that* $\widetilde{H}_{\mathrm{eff}} \not\cong \mathrm{Spin}(8)$.

    An inclusion $\mathrm{SO}(2n) \hookrightarrow \mathrm{SO}(2n+1)$ for $n \neq 4$ or $\mathrm{SU}(3) \hookrightarrow G_2$ induces an inclusion of root lattices in a unique way by Lemmas 3.22 and 3.23. It follows that the maps $R\widetilde{K}^\pm_{\mathrm{eff}} \rightrightarrows R\widetilde{H}_{\mathrm{eff}}$ have the same image, so we have a sphere bundle configuration.

  - *Suppose that* $\widetilde{H}_{\mathrm{eff}} \cong \mathrm{Spin}(8)$.

    * If the inclusions of root lattices induced by $\widetilde{H}_{\mathrm{eff}} \rightrightarrows \widetilde{K}^\pm_{\mathrm{eff}}$ are both standard, then as in the previous item, we have a sphere bundle configuration.

    * Otherwise our B_4 lattices are both of those described in Lemma 3.23 and so together span an F_4 lattice, and the intersection $RK^- \cap RK^+$ in $RH = R\mathrm{Spin}(8)$ is RF_4. By Lemma 3.24, then, $R\mathrm{Spin}(8)$ is free over RF_4 on $1152/192 = 6$ elements and each $R\mathrm{Spin}(9)$ is free on $1152/384 = 3$ elements, so by arithmetic,

    $$\frac{R\widetilde{H}_{\mathrm{eff}}}{R\widetilde{K}^-_{\mathrm{eff}} + R\widetilde{K}^+_{\mathrm{eff}}} \cong RF_4 \cong R\widetilde{K}^-_{\mathrm{eff}} \cap R\widetilde{K}^+_{\mathrm{eff}}.$$

- *Suppose that* $\widetilde{H}^-_{\mathrm{eff}} = \widetilde{H}^+_{\mathrm{eff}}$.

  - *Suppose that neither of* $H^\pm_{\mathrm{eff}}$ *is isomorphic to* $\mathrm{SO}(4)$.

    The assumption implies $H^\pm_{\mathrm{eff}}$ and hence the single or double covers $\widetilde{H}^\pm_{\mathrm{eff}}$ are simple. Since H is a product of simply-connected groups and SO(odd) factors, and since subgroups $\widetilde{K}^\pm_{\mathrm{eff}} \leqslant K^\pm$ singly or doubly covering $K^\pm_{\mathrm{eff}}$ under $\alpha^\pm$ cannot be chosen such that $\widetilde{H}^\pm_{\mathrm{eff}} = H \cap \widetilde{K}^\pm_{\mathrm{eff}}$ agree, we must have $\widetilde{H}^-_{\mathrm{eff}} \cap \widetilde{H}^+_{\mathrm{eff}} = 1$. Thus there exists a factorization

    $$H = \widetilde{H}^-_{\mathrm{eff}} \times \widetilde{H}^+_{\mathrm{eff}} \times \Pi$$

    for Π a product of totally ineffective factors contained in $K^- \cap K^+$. Since $\operatorname{rk} \widetilde{K}^\pm_{\mathrm{eff}} = \operatorname{rk} \widetilde{H}^\pm_{\mathrm{eff}}$ and the groups $\widetilde{H}^\pm_{\mathrm{eff}}$ are simple, it follows

    $$\widetilde{K}^+_{\mathrm{eff}} \cap \widetilde{H}^-_{\mathrm{eff}} = 1 = \widetilde{K}^-_{\mathrm{eff}} \cap \widetilde{H}^+_{\mathrm{eff}},$$

    and as $H = \widetilde{H}^-_{\mathrm{eff}} \times \widetilde{H}^+_{\mathrm{eff}} \times \Pi$ is contained in both groups $K^\pm$, they must admit abstract decompositions

    $$K^- \cong \widetilde{K}^-_{\mathrm{eff}} \times \widetilde{H}^+_{\mathrm{eff}} \times \Pi,$$
    $$K^+ \cong \widetilde{H}^-_{\mathrm{eff}} \times \widetilde{K}^+_{\mathrm{eff}} \times \Pi$$

    respecting the inclusions $\widetilde{H}^\pm_{\mathrm{eff}} \hookrightarrow \widetilde{K}^\pm_{\mathrm{eff}}$. Thus we may factor out $R\Pi$ and afterwards have a join configuration.

∘ *Suppose at least one of $H^{\pm}_{\text{eff}}$ is isomorphic to* $\mathrm{SO}(4)$.

We may suppose without loss of generality that it is H^{+}_{eff} which is isomorphic to $\mathrm{SO}(4)$, so that $\widetilde{H}^{+}_{\text{eff}} \cong \mathrm{Spin}(4) \cong \mathrm{Sp}(1)^2$ and $\widetilde{K}^{+}_{\text{eff}} \cong \mathrm{Spin}(5) \cong \mathrm{Sp}(2)$. Since $\widetilde{H}^{+}_{\text{eff}}$ and $\widetilde{H}^{-}_{\text{eff}}$ are both direct factors of the semisimple group H and we have assumed that $\widetilde{H}^{-}_{\text{eff}} \neq \widetilde{H}^{+}_{\text{eff}}$, we have a dichotomy based on whether $\widetilde{H}^{-}_{\text{eff}}$ shares 0 or 1 of the $\mathrm{Sp}(1)$ factors of $\widetilde{H}^{+}_{\text{eff}}$.

∗ *Suppose no* $\mathrm{Sp}(1)$ *factor of $\widetilde{H}^{+}_{\text{eff}}$ lies in $\widetilde{H}^{-}_{\text{eff}}$.*

Then $\widetilde{H}^{+}_{\text{eff}} \leqslant \Pi^-$, so we have

$$H \cong \widetilde{H}^{-}_{\text{eff}} \times \Pi^- \cong \widetilde{H}^{-}_{\text{eff}} \times \widetilde{H}^{+}_{\text{eff}} \times L$$

for some direct complement L with $\Pi^- \cong \widetilde{H}^{+}_{\text{eff}} \times L$. It follows

$$K^- \cong \widetilde{K}^{-}_{\text{eff}} \times \widetilde{H}^{+}_{\text{eff}} \times L.$$

On the other hand, the inclusion $H \hookrightarrow K^+$ factors abstractly as

$$\widetilde{H}^{-}_{\text{eff}} \times \widetilde{H}^{+}_{\text{eff}} \times L \rightarrowtail \widetilde{K}^{+}_{\text{eff}} \times \Pi^+,$$

with the image of $\widetilde{H}^{+}_{\text{eff}}$ lying in $\widetilde{K}^{+}_{\text{eff}}$, so it follows $\Pi^+ \cong \widetilde{H}^{-}_{\text{eff}} \times L$. Thus we factor out L and achieve a join configuration.

∗ *Suppose one* $\mathrm{Sp}(1)$ *factor of $\widetilde{H}^{+}_{\text{eff}}$ lies in $\widetilde{H}^{-}_{\text{eff}}$.*

Since H^{-}_{eff} is isomorphic to either $\mathrm{SU}(3)$ or $\mathrm{SO}(\text{even})$ and $\widetilde{H}^{-}_{\text{eff}}$ is a product of direct factors of $H = \mathrm{Sp}(1)^2 \times \Pi^+$, we must also have $\widetilde{H}^{-}_{\text{eff}} \cong \mathrm{Sp}(1)^2$ and $\widetilde{K}^{-}_{\text{eff}} \cong \mathrm{Sp}(2)$. Factoring out $\Pi^- \cap \Pi^+ < H$, what remains are the inclusions $\widetilde{H} \rightrightarrows \widetilde{K}^{\pm}_{\text{eff}}$, which can be identified with

$$\mathrm{Sp}(2) \times \mathrm{Sp}(1) \longleftarrow \mathrm{Sp}(1)^3 \longrightarrow \mathrm{Sp}(1) \times \mathrm{Sp}(2).$$

Then by Lemma 3.25, $R\mathrm{Sp}(3)$ is free over $R\big(\mathrm{Sp}(1)^3\big)$ on $6 = |\Sigma_3|$ elements and each of $R\widetilde{K}^{\pm}_{\text{eff}}$ is free on 3 elements, meaning

$$\frac{R\widetilde{H}_{\text{eff}}}{R\widetilde{K}^{-}_{\text{eff}} + R\widetilde{K}^{+}_{\text{eff}}} \cong RC_3 \cong R\widetilde{K}^{-}_{\text{eff}} \cap R\widetilde{K}^{+}_{\text{eff}}$$

as expected.

1. *The case exactly one of $H^{\pm}_{\text{eff}}$ is a circle*

Without loss of generality, assume that $H^{-}_{\text{eff}} \cong \mathrm{SO}(2)$ and $H^{+}_{\text{eff}} \not\cong \mathrm{SO}(2)$. As before let $\widetilde{K}^{\pm}_{\text{eff}}$ be normal virtual complements to the normal subgroups $\ker \alpha^{\pm} \lhd K^{\pm}$ and $\widetilde{H}^{\pm}_{\text{eff}} = H \cap \widetilde{K}^{\pm}_{\text{eff}}$. By our assumption on the structure of K^-, we can write

$$K^- \cong (\widetilde{K}^{-}_{\text{eff}} \times \Pi^-) \bigotimes_F A$$

for $A = Z(K^-)_0$ and Π^- a direct complement to $\widetilde{K}^{-}_{\text{eff}}$ in the commutator group $(K^-)'$, and $F \cong (\widetilde{K}^{-}_{\text{eff}} \times \Pi^-) \cap A$. Then $H \cap (\widetilde{K}^{-}_{\text{eff}} \times \Pi^-) = \widetilde{H}^{-}_{\text{eff}} \times \Pi^-$, and $K^-/H \approx S^2$, so by Lemma 3.6, we may write $H \cong (\widetilde{H}^{-}_{\text{eff}} \times \Pi^-) \otimes_F A$. Since $\widetilde{H}^{-}_{\text{eff}}$ is a circle, we have $H' = \Pi^-$.

Now $\widetilde{K}^+_{\text{eff}}$ is not isomorphic to either Spin(3) or SO(3), so $\widetilde{H}^+_{\text{eff}}$ is a closed subgroup of Π^-. By our assumption on $(K^+)'$, then, $\widetilde{K}^+_{\text{eff}}$ is a direct factor and there exists a complement $L \lhd \Pi^-$ with

$$\Pi^- \cong L \times \widetilde{H}^+_{\text{eff}},$$
$$(K^+)' \cong L \times \widetilde{K}^+_{\text{eff}}.$$

It is clear then that $K^+ = \widetilde{H}^-_{\text{eff}} \cdot (L \times \widetilde{K}^+_{\text{eff}}) \cdot A$. We have

$$\widetilde{H}^-_{\text{eff}} \cap (L \times \widetilde{K}^+_{\text{eff}}) = \widetilde{H}^-_{\text{eff}} \cap H \cap (L \times \widetilde{K}^+_{\text{eff}}) = \widetilde{H}^-_{\text{eff}} \cap (L \times \widetilde{H}^+_{\text{eff}}) = 1$$

and also

$$(\widetilde{H}^-_{\text{eff}} \times L \times \widetilde{K}^+_{\text{eff}}) \cap A = (\widetilde{H}^-_{\text{eff}} \times L \times \widetilde{K}^+_{\text{eff}}) \cap H \cap A = (\widetilde{H}^-_{\text{eff}} \times L \times \widetilde{H}^+_{\text{eff}}) \cap A = F,$$

so in fact $K^+ \cong (\widetilde{H}^-_{\text{eff}} \times L \times \widetilde{K}^+_{\text{eff}}) \otimes_F A$.

Thus we may factor out A/F and then L to obtain a join configuration.

2. *The case $H^\pm_{\text{eff}}$ are both circles*

The intersections $\Pi^\pm$ of $(K^\pm)'$ with the ineffective $\ker \alpha^\pm$ admit complements $\widetilde{K}^\pm_{\text{eff}}$ in $(K^\pm)'$ by assumption. Since $\operatorname{im} \alpha^\pm \cong \text{SO}(3)$ is simple and centerless, the centers $Z(K^\pm)$ are also contained in $\ker \alpha^\pm$. This kernel is obviously contained in the stabilizer H as well, so $\Pi^\pm = (\Pi^\pm)' \leqslant H'$. On the other hand, since the images $\alpha^\pm(H) \cong \text{SO}(2)$ are abelian, the commutator subgroup H' is contained in both of $\ker \alpha^\pm$, so $\Pi^\pm = H'$.

By the assumption on $(K^\pm)'$, we have

$$K^\pm \cong (H' \times \widetilde{K}^\pm_{\text{eff}}) \cdot Z(K^\pm)_0,$$
$$H \cong (H' \times \widetilde{H}^\pm_{\text{eff}}) \cdot Z(K^\pm)_0.$$

Now consider the torus $A := \big(Z(K^-) \cap Z(K^+)\big)_0$. Taking $\underline{H} = H'\widetilde{H}^-_{\text{eff}}\widetilde{H}^+_{\text{eff}}$, and $F = \underline{H} \cap A$, we may write $H \cong \underline{H} \otimes_F A$. If we set $\underline{K}^\pm = \underline{H}\widetilde{K}^\pm_{\text{eff}}$, then evidently $\underline{K}^\pm \cap H = \underline{H}$ and $K^\pm = \underline{K}^\pm A$. Since

$$\underline{K}^\pm \cap A = \underline{K}^\pm \cap H \cap A = \underline{H} \cap A = F,$$

we find $K^\pm \cong \underline{K}^\pm \otimes_F A$, so we may factor out A/F.

- *Suppose $A = Z(K^-)_0 = Z(K^+)_0$.*

  In this case $Z(H)/A$ is one-dimensional, so we may select $\widetilde{K}^\pm_{\text{eff}}$ in such a way that $\widetilde{H}_{\text{eff}} = \widetilde{H}^-_{\text{eff}} = \widetilde{H}^+_{\text{eff}} \cong \text{SO}(2)$. Factoring out A/F and then H' leaves a configuration $\text{SO}(2) \rightrightarrows \widetilde{K}^\pm_{\text{eff}}$ where $\widetilde{K}^\pm_{\text{eff}}$ are each SO(3) or Spin(3). Either way, the induced map $R\widetilde{K}^\pm_{\text{eff}} \longrightarrow R\text{SO}(2) = \mathbb{Z}[t]$ has image $\mathbb{Z}[t + t^{-1}]$, so we are functionally in the situation of Proposition 5.4 and in particular

  $$\frac{R\widetilde{H}_{\text{eff}}}{R\widetilde{K}^-_{\text{eff}} + R\widetilde{H}^+_{\text{eff}}} \cong \frac{\mathbb{Z}[t]}{\mathbb{Z}[t + t^{-1}]} \cong t \cdot \mathbb{Z}[t + t^{-1}]$$

  is of rank one over $\mathbb{Z}[t + t^{-1}]$.

- *Suppose* $Z(K^-)_0 \neq Z(K^+)_0$.

  Write T for the two-dimensional torus $\widetilde{H}^-_{\mathrm{eff}} \cdot \widetilde{H}^+_{\mathrm{eff}}$ in H. Then after factoring out A/F we have to deal with the inclusions of $H = H' \times T$ in $(H' \times \widetilde{K}^{\pm}_{\mathrm{eff}}) \cdot S^1$, where $\mathrm{id}_{H'}$ factors out of these inclusions but we claim nothing particular about the two inclusions $T \rightarrowtail \widetilde{K}^{\pm}_{\mathrm{eff}} \cdot S^1$. Factoring out H', we arrive at $\underline{H} = T$ and $\underline{K}^{\pm} \cong \widetilde{K}^{\pm}_{\mathrm{eff}} \otimes_F S^1$, where $|F| \leqslant 2$.

  The inclusions $T \hookrightarrow \underline{K}^{\pm}$ induce inclusions $R\underline{K}^{\pm} \cong (RT)^{\langle w_{\pm} \rangle} \hookrightarrow RT$, where $w_{\pm}$ generates $W\underline{K}^{\pm} \cong \mathbb{Z}/2$. Identifying RT^2 with the group ring $\mathbb{Z}\mathrm{X}$ of the character group $\mathrm{X} = \mathrm{X}(T) = \mathrm{Hom}(T, S^1)$, these can be seen as induced by two reflections of the vector space $\mathfrak{t}^\vee \cong \mathbb{R}^2$ which preserve the integer lattice $\mathrm{X}(T) \cong \mathbb{Z}^2$. Under this identification $W = \langle w_-, w_+ \rangle$ becomes a dihedral subgroup D_{2k} of $\mathrm{GL}(2, \mathbb{Z})$. These are classified: they can only be D_4, D_6, D_8, D_{12} and are conjugate to the standard presentations for the Weyl groups of types $D_2 = A_1 \times A_1$, A_2, $B_2 = C_2$, and G_2 as well as a second $D_6 < WG_2$ not generated by root reflections, which hence does not occur [Tah71, Prop. 1][Mack96]. The root lattice Q_W and weight lattice P_W corresponding to reflection groups W of this type in $\mathbb{R}^2$ are unique (up to equivariant isomorphism) and there are examples, most of which we produce immediately following the present argument, showing any intermediate lattice between Q_W and P_W occurs as X for some cohomogeneity-one action.

  In all of these cases, we need to see

$$\Theta := \frac{RT}{(RT)^{\langle w_- \rangle} + (RT)^{\langle w_+ \rangle}}$$

  is a free cyclic module over $(RT)^W$. One is tempted is to use Theorem 3.21, but it can happen that RT is not free over $(RT)^W$. Instead our answer comes from the Stiefel diagram. The ring RT is free on the $\mathbb{Z}$-basis X. Quotienting by $(RT)^{\langle w_- \rangle} + (RT)^{\langle w_+ \rangle}$, annihilates $\mathrm{X}^{\langle w_- \rangle}$ and $\mathrm{X}^{\langle w_- \rangle}$ and induces relations

$$\begin{aligned} w_-\theta &\equiv -\theta \qquad\qquad \text{for } \theta \notin \mathrm{X}^{\langle w_- \rangle}, \\ w_+\theta &\equiv -\theta \qquad\qquad \text{for } \theta \notin \mathrm{X}^{\langle w_+ \rangle}, \end{aligned}$$

  since $\theta + w_-\theta \in (RT)^{\langle w_- \rangle}$ and $\theta + w_+\theta \in (RT)^{\langle w_+ \rangle}$. It follows Θ admits a $\mathbb{Z}$-basis given by those characters of T lying in the interior C of a fundamental domain.[16]

  On the other hand, $(RT)^W$ is spanned by orbit sums $S\theta = \sum_{w \in W/\mathrm{Stab}\,\theta} w\theta$. These are indexed by W-orbits of X, of which there is precisely one per character θ in the closed fundamental domain $\overline{C}$. Drawing out the diagrams, one checks for each lattice type that there is a minimal strongly dominant integral weight λ_0, which makes $\theta \longleftrightarrow \theta \cdot \lambda_0$ a bijection $\overline{C} \cap \mathrm{X} \longleftrightarrow C \cap \mathrm{X}$.[17] Recall that if X is given the partial order determined by setting $\sigma \geqslant \theta$ just when θ lies in the convex hull of the orbit $W \cdot \sigma$, then given $\sigma, \theta \in \mathrm{X} \cap \overline{C}$, the difference $S(\sigma\theta) - S\sigma \cdot S\theta$ is a sum of terms of lower order [Adams69, Prop. 6.36]. If we filter Θ with respect to this order, then it follows the $(RT)^W$-module structure on the associated graded

[16] The notation C is meant to suggest a Weyl chamber, even though our dihedral group is just a group of symmetries of a $\mathbb{Z}^2$ lattice, not *a priori* the Weyl group of anything, because the same reasoning goes through.

[17] If X is the lattice spanned by the fundamental weights dual to the simple roots of the root system for W, so that half the sum of positive roots is an integral weight ρ, then [Adams69, Lem. 5.58] we have $\rho = \lambda_0$. But these are not all the cases.

module $\operatorname{gr}\Theta$ is given by $S\sigma \cdot \overline{\theta\lambda_0} = \overline{(\sigma\theta)\lambda_0}$, so Θ is the free cyclic $(RT)^W$-module generated by λ_0 as claimed. □

Remark 5.11. It is interesting to note that all of the exceptional cases occur as the "degree-generating actions" tabulated by Püttmann [Püt09, §5.2][GrWZ08, Table E, p. 105]. The actions of F_4 on S^{25} and $\mathrm{Spin}(3)$ on S^{13} already came up in the "no circular isotropy" case, and the others are among the "two circles" cases, as per the following examples.

Example 5.12. The dihedral group D_4, a Coxeter group of Killing–Cartan type D_2, is realized as the Weyl group of a cohomogeneity-one action with $H \cong T^2$ as follows. One has an isomorphism $\mathrm{SO}(4) \cong \mathrm{Spin}(3) \underset{\mathbb{Z}/2}{\otimes} \mathrm{Spin}(3)$ and can consider the diagram

$$G \cong \mathrm{SO}(4), \quad K^- = \mathrm{Spin}(3) \underset{\mathbb{Z}/2}{\otimes} \mathrm{Spin}(2), \quad K^+ = \mathrm{Spin}(2) \underset{\mathbb{Z}/2}{\otimes} \mathrm{Spin}(3), \quad H = \mathrm{Spin}(2) \underset{\mathbb{Z}/2}{\otimes} \mathrm{Spin}(2) = T.$$

Write $\widetilde{T} = \mathrm{Spin}(2) \times \mathrm{Spin}(2)$ and $R\widetilde{T} = \mathbb{Z}[s, t, s^{-1}t^{-1}]$. Then $W = W\mathrm{SO}(4) \cong S_2 \times \{\pm 1\}$. Since $\mathrm{SO}(4)$ is not simply-connected [Ste75], we see $RT = \mathbb{Z}[s^{\pm 1}t^{\pm 1}]$ is not free over

$$R\mathrm{SO}(4) \cong (RT)^W \cong \mathbb{Z}[s + s^{-1} + t + t^{-1}, st + s^{-1}t^{-1}, s^{-1}t + st^{-1}],$$

illustrating the proof of the $H^{\pm}_{\mathrm{eff}} = \mathrm{SO}(2)$ case in Theorem 0.4 cannot be run through Theorem 3.21 in all cases.

Instead considering the two-fold covers inside $G = \mathrm{Spin}(4) \cong \mathrm{Spin}(3)^2$, one obtains a Weyl group of type D_2 again, but now $R\widetilde{T} = \mathbb{Z}[s, t, s^{-1}t^{-1}]$ is free over

$$R\mathrm{Spin}(4) = (RT)^W \cong \mathbb{Z}[s + s^{-1} + t + t^{-1}, st + s^{-1}t^{-1}],$$

and one can apply Theorem 3.21 again. The space acted on is $S^2 * S^2 \approx S^5$.

We leave it to the reader to construct an analogous example with $G = \mathrm{SO}(3) \times \mathrm{SO}(3)$.

Example 5.13. The dihedral group D_6, a Coxeter group of Killing–Cartan type A_2, is realized as the Weyl group of a cohomogeneity-one action with $H \cong T^2$ as follows. Consider the diagram

$$G = \mathrm{U}(3), \qquad K^- = \mathrm{U}(2) \times \mathrm{U}(1), \qquad K^+ = \mathrm{U}(1) \times \mathrm{U}(2), \qquad H = \mathrm{U}(1)^3.$$

In the notation of the proof of Theorem 0.4, the irrelevant torus $A = Z\big(\mathrm{U}(3)\big) \cong S^1$ is the group of diagonal matrices and $F \cong \langle e^{2\pi i/3} \rangle$. After factoring out A/F, one has the corresponding subgroups of $\mathrm{SU}(3)$, and the manifold is S^7. The reduced $\underline{K^{\pm}}$ are both isomorphic to $\mathrm{U}(2)$, and one has $W = W\mathrm{SU}(3) = \Sigma_3$ with $w_- = (1\ 2)$ and $w_+ = (2\ 3)$. Since $\mathrm{SU}(3)$ is simply-connected and it is easy to check the coset condition applies, one could also apply Theorem 3.21.

Example 5.14. The dihedral group D_8, a Coxeter group of Killing–Cartan type BC_2, is realized as the Weyl group of a cohomogeneity-one action with $H \cong T^2$ as follows. Consider the diagram

$$G \cong \mathrm{SO}(5), \qquad K^- = \mathrm{U}(2) \times \{1\}, \qquad K^+ = \mathrm{SO}(2) \times \mathrm{SO}(3), \qquad H = \mathrm{SO}(2) \times \mathrm{SO}(2) \times \{1\} = T,$$

where all subgroups are block-diagonal, $\mathrm{U}(2) \oplus \{1\}$ being embedded in the block-diagonal $\mathrm{SO}(4) \oplus \{1\}$ in the expected manner. Then $WG \cong \Sigma_2 \ltimes \{\pm 1\}^2$ is a Coxeter group of type B_2 acting on $\mathfrak{t}^2$ as the dihedral group D_8 and is generated by $w_- = \big((1\ 2), 1, 1\big)$ and $w_+ = (\mathrm{id}, 1, -1)$. Theorem 3.21 does not apply as stated, as $\mathrm{SO}(5)$ is not simply-connected, but the relevant part of Steinberg's proof [Ste75] only requires that $R\mathrm{SO}(5)$ be polynomial, which it is, and one can check the coset condition holds.

One can also consider the cover

$$G = \mathrm{Spin}(5) = \mathrm{Sp}(2), \qquad K^- = \mathrm{U}(2), \qquad K^+ = \mathrm{U}(1) \oplus \mathrm{Sp}(1), \qquad H = \mathrm{U}(1) \oplus \mathrm{U}(1) = T,$$

which generates the same W.

Example 5.15. The dihedral group D_{12}, a Coxeter group of Killing–Cartan type G_2, is realized as the Weyl group of a cohomogeneity-one action with $H \cong T^2$ as follows. Consider the adjoint action of the compact exceptional group G_2 on its Lie algebra $\mathfrak{g}_2 \cong \mathbb{R}^{14}$. This restricts to an action on the unit sphere S^{13} under the norm induced by the Killing form, and the orbits are given by the intersection of S^{13} with a Weyl chamber in the Lie algebra $\mathfrak{t}^2$ of a maximal torus, cutting out an arc of the unit circle $S^1 \subsetneq \mathfrak{t}^2$ of angle $\pi/6$. The principal isotropy group fixing a point on the interior of the arc is T^2 itself and the singular isotropies fixing the endpoints are two nonconjugate copies of $\mathrm{U}(2)$ [Miy01]. The reflections $w_\pm$ generate the dihedral group $WG_2 = D_{12}$. As G_2 is simply-connected, one can check the coset condition and apply Theorem 3.21 again.

6. Equivariant formality

In this final section, we let $G \curvearrowright M$ be a cohomogeneity-one action with M/G a closed interval as in the first fork 0.2(a) of Mostert's dichotomy 0.1 and use the structure theorems for $K^*_G(M)$ in the previous two sections and the representation theory of Section 3 to characterize equivariant formality of such actions.

Recall that ***K-theoretic equivariant formality*** means surjectivity of the map $K^*_G(M) \longrightarrow K^*(M)$ forgetting the G-equivariant structure on a complex vector bundle. This condition, first studied by Matsunaga and Minami [MatM86][18] is stronger than the condition that $K^*_G(M;\mathbb{Q}) \longrightarrow K^*(M;\mathbb{Q})$ be surjective, which Fok [Fok19] named *rational K-theoretic equivariant formality* and showed is equivalent to cohomological *equivariant formality* in the traditional sense [GorKM98] that the restriction $H^*_G(M;\mathbb{Q}) \longrightarrow H^*(M;\mathbb{Q})$ along the fiber inclusion in the Borel fibration $M \to M_G \to BG$ be surjective. Goertsches and Mare [GoeM14, Cor. 1.3] showed a cohomogeneity-one action of a compact, connected Lie group G on a smooth closed manifold M with orbit space an interval is equivariantly formal if and only if $\operatorname{rk} G = \max\{\operatorname{rk} K^-, \operatorname{rk} K^+\}$, so the same holds of rational K-theoretic equivariant formality and the rank equation is a necessary condition for K-theoretic equivariant formality over the integers. The converse also holds, at least with the standard restriction on fundamental groups.

Theorem 6.1. *Consider a cohomogeneity-one action of a compact, connected Lie group G with $\pi_1(G)$ torsion-free on a smooth closed manifold M such that the orbit space M/G is an interval and the commutator subgroups of the exceptional isotropy groups $K^\pm$ are the products of simply-connected groups and* $\mathrm{SO}(\mathrm{odd})$ *factors. Then the action is K-theoretically equivariantly formal if and only if* $\operatorname{rk} G = \max\{\operatorname{rk} K^-, \operatorname{rk} K^+\}$.

Proof. We consider the Hodgkin–Künneth spectral sequence [Hodgkin, Intro., Cor. 1, p. 6] for the left multiplication G-action on $X = G$ and the given action on $Y = M$, a $(\mathbb{Z} \times \mathbb{Z}/2)$-graded

[18] though Hodgkin had already dubbed the map "forgetful" [Hodgkin, p. 72]

left–half-plane spectral sequence which starts at

$$E_2^{*,*} = \mathrm{Tor}_{RG}^{*,*}(K_G^* X, K_G^* Y) = \mathrm{Tor}_{RG}^{*,*}(\mathbb{Z}, K_G^* M)$$

and, given the hypothesis on $\pi_1 G$, converges to

$$K_G^*(X \times Y) = K_G^*(G \times M) \cong K^*(M).$$

The forgetful map $K_G^*(M) \longrightarrow K^*(M)$ we wish to show is surjective can be identified [Hodgkin, Prop. 9.1, p. 71] with the edge map

$$K_G^*(M) \twoheadrightarrow \mathbb{Z} \underset{RG}{\otimes} K_G^*(M) = E_2^{0,*} \hookrightarrow E_\infty^{0,\bullet}.$$

In each case we will verify the groups $\mathrm{Tor}_{RG}^{\leqslant -1}(\mathbb{Z}, K_G^* M) = 0$ vanish, showing the spectral sequence collapses and the edge map is a surjection. We will repeatedly use the following facts. First, if K/H is an odd-dimensional sphere, then $\mathrm{rk}\, K = 1 + \mathrm{rk}\, H$, while if K/H is an even-dimensional sphere, then $\mathrm{rk}\, K = \mathrm{rk}\, H$. Second [AtH61, Thm. 3.6], for Γ closed and connected of full rank in G we have $K^1(G/\Gamma) = 0$ and $K^0(G/\Gamma)$ free abelian (of rank $|WG|/|W\Gamma|$). Third [GonZ17, (7), p. 19], the groups $\mathrm{Tor}_{RG}^{\leqslant p}(\mathbb{Z}, R\Gamma)$ vanish for $\Gamma \leqslant G$ closed and connected with $\mathrm{rk}\, G - \mathrm{rk}\, \Gamma < |p|$, so that particularly $\mathrm{Tor}_{RG}^{\leqslant -2}(\mathbb{Z}, R\Gamma)$ vanishes for $\Gamma \in \{K^\pm, H\}$.

Suppose $\mathrm{rk}\, G = \mathrm{rk}\, H + 1$.

In these cases we know that one of $K^\pm$ has rank greater than that of H, and our hypothesis on $K^\pm$ implies that $R(K^\pm)'$ is polynomial [Ste75], so the corresponding restriction $RK^\pm \longrightarrow RH$ is surjective by Propositions 3.10 and 3.13 and the Mayer–Vietoris sequence of Theorem 4.1 shows $K_G^1(M)$ vanishes, leaving a short exact sequence of RG-modules $K_G^0(M) \rightarrowtail RK^- \times RK^+ \twoheadrightarrow RH$. Applying the derived exact sequence of the functor $\mathbb{Z} \otimes_{RG} -$, we find $\mathrm{Tor}_{RG}^{\leqslant -2}(\mathbb{Z}, K_G^* M)$ vanishes as above. Since in fact the E_2 page is only inhabited by $E_2^{0,0}$ and $E_2^{-1,0}$, we know the former of these is $K^0(M)$ and the latter $K^1(M)$. Thus the forgetful map will be surjective if and only if also $\mathrm{Tor}_{RG}^{-1}(\mathbb{Z}, K_G^0 M) = K^1(M) = 0$. Using the Mayer–Vietoris sequence of the standard cover, we must show $K^0(G/K^-) \oplus K^0(G/K^+) \longrightarrow K^0(G/H)$ is surjective and $K^1(G/K^-) \oplus K^1(G/K^+) \longrightarrow K^1(G/H)$ injective.

For surjectivity, assume without loss of generality that $\mathrm{rk}\, G = \mathrm{rk}\, K^+$, so that $K^1(G/K^+)$ is zero and $K^0(G/K^+)$ is free abelian; in particular, then the Atiyah–Hirzebruch spectral sequence $H^*(G/K^+) \implies K^*(G/K^+)$ collapses. There is an evident bundle map

$$\begin{array}{ccc} K^+/H & \longrightarrow & * \\ \downarrow & & \downarrow \\ G/H & \longrightarrow & G/K^+ \\ \downarrow & & \| \\ G/K^+ & = & G/K^+ \end{array}$$

inducing a map of Atiyah–Hirzebruch–Leray–Serre spectral sequences. We have just seen the right spectral sequence collapses, and the map then shows all differentials out of the zero row of the left spectral sequence must vanish as well. Particularly this means that the row $E_\infty^{*,0}$ is a quotient of $E_2^{*,0} = K^*(G/K^+)$; and since K^+/H is an odd-dimensional sphere, $K^*(K^+/H)$ is an exterior algebra $\Lambda[z]$ on one generator $z \in K^1(K^+/H)$, so that

$$E_2 = H^*\big(G/H; K^*(K^+/H)\big) \cong H^*(G/H) \otimes \Lambda[z].$$

Since each diagonal thus contains only one nonzero entry, we have $E_\infty \cong K^*(G/H)$ as groups and thus, since odd columns are zero, $E_\infty^{*,0} \cong K^0(G/H)$. This is a quotient of the row $E_2^{*,0} \cong H^*(G/K^+)$, so the collapse $H^*(G/K^+) \cong K^0(G/K^+)$ of the Atiyah–Hirzebruch spectral sequence on the right shows $K^0(G/K^+) \longrightarrow K^0(G/H)$ is surjective.

Injectivity is obvious if $K^1(G/K^\pm) = 0$, so now assume as well that $\operatorname{rk} K^- = \operatorname{rk} H = \operatorname{rk} G - 1$. We consider the map of Hodgkin–Künneth spectral sequences corresponding to $X = G$ and $G/H = Y \to Y' = G/K^-$. These are concentrated in the 0-row and again by the vanishing of $\operatorname{Tor}^{\leqslant -2}$, the spectral sequences both collapse at E_2, so the map $K^1(G/K^-) \longrightarrow K^1(G/H)$ may be identified with the map $\operatorname{Tor}^{-1}_{RG}(\mathbb{Z}, RK^-) \longrightarrow \operatorname{Tor}^{-1}_{RG}(\mathbb{Z}, RH)$. But as K^-/H is an even-dimensional sphere by assumption, Proposition 3.20 shows RH is free of rank two over RK^-, so one has a short exact sequence $RK^- \rightarrowtail RH \twoheadrightarrow RK^-$. Applying the derived exact sequence of $\mathbb{Z} \otimes_{RG} -$ and the vanishing of Tor^{-2}, we see $\operatorname{Tor}^{-1}_{RG}(\mathbb{Z}, RK^-) \longrightarrow \operatorname{Tor}^{-1}_{RG}(\mathbb{Z}, RH)$ is injective as claimed.

Suppose $\operatorname{rk} G = \operatorname{rk} H$.

Since $K^1(G/K^\pm) = 0 = K^1(G/H)$ in this situation, the sequence of Theorem 2.11 separates into the two short exact sequences

$$0 \to K^0_G(M) \longrightarrow RK^- \times RK^+ \longrightarrow B \to 0,$$

$$0 \to B \longrightarrow RH \longrightarrow K^1_G(M) \to 0$$

of RG-modules. From the vanishing of $\operatorname{Tor}^{\leqslant -2}$ we get RG-module isomorphisms

$$\operatorname{Tor}^{-n-2}_{RG}(\mathbb{Z}, K^1_G M) \cong \operatorname{Tor}^{-n-1}_{RG}(\mathbb{Z}, B) \cong \operatorname{Tor}^{-n}_{RG}(\mathbb{Z}, K^0_G M) \qquad (n \geqslant 1),$$

and from Theorem 0.4 we also have an RG-module isomorphism $K^0_G(M) \cong K^1_G(M)$, so the higher Tors are 2-periodic. But $\mathbb{Z}$ has finite projective dimension over RG (indeed, the Koszul algebra $RG \otimes K^*G$ is a resolution of length $\operatorname{rk} G$), so these higher Tors vanish. □

Remark 6.2. The last sentence in this proof, the observation it concludes the proof, and the request for such a result in the first place are all due to Marcus Zibrowius.

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