The Cohomology of Homogeneous Spaces
and related topics

by

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Thesis

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VITA:

Joel L. Wolf was born on January 29, 1948, in New York. He attended Stuyvesant High School. In 1968 he received his Sc.B. from the Massachusetts Institute of Technology.

While at Brown he has had a N.A.S.A. fellowship; he has also taught for three years. Next year he will be a Benjamin Pierce Assistant Professor at Harvard University.

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To Cathy

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0. INTRODUCTION:

Consider a differentiable fibre bundle

$$\sigma = (E, \pi, X, G/H, G),$$

where G is a compact, connected Lie group and H a compact, connected subgroup of G, E and X are differentiable manifolds, and $\pi: E \to X$ is a differentiable map. One would like to compute the cohomology of the total space E in terms of the cohomology of the base space X, and certain algebraic invariants of the imbedding of H into G. Specifically, there exists a universal bundle

$$\sigma(G,H) = (BH, f, BG, G/H, G)$$

and a classifying diagram

$$\begin{array}{ccc} G/H & = & G/H \\ \downarrow & & \downarrow \\ E & \longrightarrow & BH \\ \downarrow & & \downarrow f \\ X & \xrightarrow[g]{} & BG \end{array}$$

One would like to obtain some sort of isomorphism

$$H^*(E;K) \approx \operatorname{tor}_{H^*(BG;K)}(H^*(X;K), H^*(BH;K)),$$

where $H^*(X;K)$ is regarded as a right $H^*(BG;K)$-module via the multiplicative map g^* and $H^*(BH;K)$ is regarded as a left $H^*(BG;K)$-module via the multiplicative map f^*.

One does have, by results of Eilenberg and Moore [11][12], an algebra isomorphism

$$H^*(E;K) \approx \mathrm{Tor}_{C^*(BG;K)}(C^*(X;K),C^*(BH;K)),$$

where $C^*(X;K)$ is regarded as a right differential $C^*(BG;K)$-module via the multiplicative map $g^\#$ and $C^*(BH;K)$ is regarded as a left differential $C^*(BG;K)$-module via the multiplicative map $f^\#$.

In a portion of his Princeton University thesis, Baum [1] gave an elegant partial answer to the question above. Considering the special case where the coefficient field K is the reals $\mathbf{R}$ and the base space X is a point $*$, Baum showed...

<u>THEOREM A</u>: Let G be a compact, connected Lie group and H a compact, connected subgroup of G. Form the homogeneous space G/H. Then, regarding $\mathbf{R}$ as a right $H^*(BG;\mathbf{R})$-module via augmentation and $H^*(BH;\mathbf{R})$ as a left $H^*(BG;\mathbf{R})$-module via the natural map $f^*:H^*(BG;\mathbf{R}) \to H^*(BH;\mathbf{R})$, we have an algebra isomorphism

$$H^*(G/H;\mathbf{R}) \approx \mathrm{tor}_{H^*(BG;\mathbf{R})}(\mathbf{R},H^*(BH;\mathbf{R})).$$

Under the hypotheses of Theorem A it follows that $H^*(BG;\mathbf{R})$ is a polynomial algebra and $C^*(BG;\mathbf{R})$ is graded commutative. Using these facts, Baum constructs a multiplicative homology isomorphism

$$\theta:H^*(BG;\mathbf{R}) \to C^*(BG;\mathbf{R})$$

and similarly a multiplicative homology isomorphism

$$\phi: H^*(BH;\mathbf{R}) \to C^*(BH;\mathbf{R}).$$

Then he makes use of various naturality properties of Tor to pass from $H^*(G/H;\mathbf{R}) \approx \mathrm{Tor}_{C^*(BG;\mathbf{R})}(\mathbf{R}, C^*(BH;\mathbf{R}))$ to $\mathrm{tor}_{H^*(BG;\mathbf{R})}(\mathbf{R}, H^*(BH;\mathbf{R}))$.

In this thesis we generalize Theorem A in two distinct directions. First, holding the base space to be a point, we generalize the coefficient field. Second, holding the coefficient field to be $\mathbf{R}$ (or the rationals $\mathbf{Q}$, in light of recent work by Sullivan), we generalize the base space.

In Chapter I we describe the various introductory material which will be needed later. I. 1 contains algebraic preliminaries. I. 2 contains geometric preliminaries.

In Chapter II we prove the following theorem on the cohomology of homogeneous spaces...

<u>THEOREM B</u>: Let G be a compact, connected Lie group and H a compact, connected subgroup of G. Form the homogeneous space G/H. Suppose that either

(i). K has characteristic 0,

or

(ii). K has characteristic p, and $H_*(G;K)$, $H_*(H;K)$ have no p-torsion.

Then, regarding K as a right $H^*(BG;K)$-module via augmentation, and $H^*(BH;K)$ as a left $H^*(BG;K)$-module via the natural map $f^*: H^*(BG;K) \to H^*(BH;K)$, we have a module isomorphism

$$H^*(G/H;K) \approx \mathrm{tor}_{H^*(BG;K)}(K, H^*(BH;K)).$$

The proof is in the spirit of Baum's proof of Theorem A.

Under the hypotheses of Theorem B it follows that $H^*(BG;K)$ is a polynomial algebra; but $C^*(BG;K)$ is not, in general, graded commutative. $C^*(BG;K)$ is, however, homotopy commutative (via $\cup_1$-products) in a very strong way. Using these facts, we construct a strongly homotopy multiplicative (shm) homology isomorphism

$$\{\theta_1, \theta_2, \theta_3, \ldots\}: H^*(BG;K) \to C^*(BG;K).$$

Using the concept of shm (due to Clark [8], Stasheff [21][22], and Stasheff and Halperin [23]) we extend the notion of torsion products to strongly homotopy modules. Then we use a result of Baum [1][2] to reduce to the case of G and a maximal torus T of H, a result of May [16] and Gugenheim and May [13] which shows the existence of a multiplicative homology isomorphism

$$\alpha: C^*(BT;K) \to H^*(BT;K)$$

which annihilates $\cup_1$-products, and various naturality properties of this new TOR to pass from $H^*(G/H;K) \approx$ $\approx \mathrm{Tor}_{C^*(BG;K)}(K, C^*(BH;K))$ to $\mathrm{tor}_{H^*(BG;K)}(K, H^*(BH;K))$.

II. 1 contains the algebraic material on tor, Tor, and TOR, described in terms of the two-sided bar construction. II. 2 gives a description of the shm map from $H^*(BG;K)$ to $C^*(BG;K)$ in terms of $\cup_1$-products. II. 3 contains the proof of Theorem B.

Theorem B has been of considerable interest to a number of mathematicians. Among those who have made significant contributions are: Baum [1][2], A. Borel [4], H. Cartan [5], Gugenheim and May [13], Husemoller, Moore and Stasheff [14], May [16], Munkholm [17][18], Stasheff [21][22], and Stasheff and Halperin [23]. Several of the above have announced proofs of a more or less general theorem along the lines of Theorem B.

In Chapter III we prove the following theorem on the real and rational cohomology of differentiable fibre bundles...

<u>THEOREM C</u>: Let

$$\sigma = (E,\pi,X,G/H,G)$$

be a differentiable fibre bundle with X a homogeneous space formed as the quotient G'/H' of a compact, connected Lie group G' by a compact, connected subgroup H' of deficiency 0 in G'. Suppose that either K is the reals **R** or the rationals **Q**. Then, regarding H*(X;K) as a right H*(BG;K)-module via the natural map $g^*:H^*(BG;K)\to H^*(X;K)$ and H*(BH;K) as a left H*(BG;K)-module via the natural map $f^*:H^*(BG;K)\to H^*(BH;K)$, we have an algebra isomorphism

$$H^*(E;K) \approx \mathrm{tor}_{H^*(BG;K)}(H^*(X;K),H^*(BH;K)).$$

The proof is again in the spirit of Baum's proof of Theorem A. Under the hypotheses of Theorem C it follows that C*(X;K) is graded commutative; but H*(X;K) is not, in general, a polynomial algebra. However, utilizing various Koszul constructions involving the cohomology and cochains of G' and of H', we construct a collection of multiplicative homology isomorphisms relating H*(X;K) and C*(X;K). Then we make use of various naturality properties of Tor to pass from $H^*(E;K) \approx \mathrm{Tor}_{C^*(BG;K)}(C^*(X;K), C^*(BH;K))$ to $\mathrm{tor}_{H^*(BG;K)}(H^*(X;K), H^*(BH;K))$.

III. 1 contains the algebraic material on tor and Tor, now described in terms of the two-sided Koszul construction. III. 2 contains the proof of Theorems A and C.

I. PRELIMINARIES

I. 1. ALGEBRA:

Fix K to be a commutative ring with unit.

A GRADED MODULE A over K will be a sequence $\{A_i \mid i = 0,1,2,\dots\}$ of K-modules. An element $a \in A_m$ is also considered to be an element of DEGREE m in A. So if $k \in K$ and $a, b \in A$ have degree m, then ka and a + b are also elements of degree m in A.

A BIGRADED MODULE A over K will be a doubly-indexed sequence $\{A_{i,j} \mid i = 0,1,2,\dots;\ j = 0,1,2,\dots\}$ of K-modules. An element $a \in A_{m,n}$ is also considered to be an element of BIDEGREE (m,n) in A. If A is a bigraded module over K, we can form the ASSOCIATED graded module over K, also denoted by A, by setting $A_k = \bigoplus_{i+j=k} A_{i,j}$.

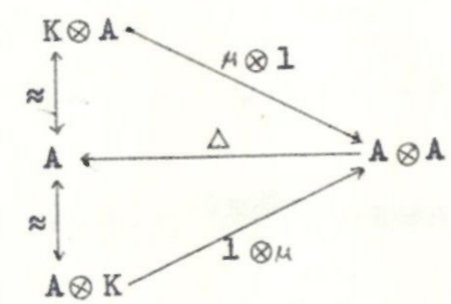
If A and B are graded modules over K, then a K-MODULE HOMOMORPHISM $f: A \rightarrow B$ of DEGREE p is a sequence $\{f_i \mid i = 0,1,2,\dots\}$ of K-module homomorphisms $f_i: A_i \rightarrow B_{i+p}$. If $a \in A$ has degree m, then f(a) denotes an element in B of degree m + p. The KERNEL of f is a graded module over K defined by $(\mathrm{Ker}(f))_m = \mathrm{Ker}(f_m)$. The IMAGE of f is a graded module over K defined by $(\mathrm{Im}(f))_m = \mathrm{Im}(f_{m-p})$.

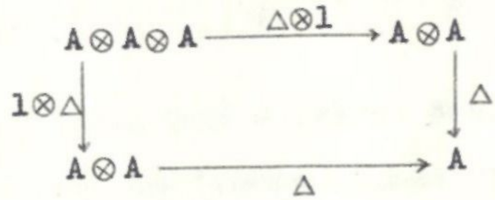
If A is a graded module over K, then a SUBMODULE B of A is a graded module over K such that B_i is a submodule of A_i for each i = 0,1,2,... For example, if $f: A \rightarrow B$ is a K-module homomorphism, then Ker(f) is a submodule of A and Im(f) is a submodule of B. If a graded module B over

K is a submodule of a graded module A over K, then the QUOTIENT graded module A/B over K is defined by $(A/B)_m = A_m/B_m$.

If A and B are graded modules over K, then $A \otimes B$ is the graded module over K defined by $(A \otimes B)_k = \bigoplus_{i+j=k} A_i \otimes B_j$. If $a \in A$ has degree m and $b \in B$ has degree n, then $a \otimes b$ denotes an element of $A \otimes B$ of degree m + n. By considering K to be the trivially graded module over K which is K in degree 0 and 0 in all other degrees, $A \otimes K$ and $K \otimes A$ are canonically isomorphic to A. If A_1, A_2, B_1, B_2 are graded modules over K, and $f_1: A_1 \to B_1$ and $f_2: A_2 \to B_2$ are K-module homomorphisms, then $f_1 \otimes f_2: A_1 \otimes A_2 \to B_1 \otimes B_2$ is the K-module homomorphism defined by $f_1 \otimes f_2(a_1 \otimes a_2) = f_1(a_1) \otimes f_2(a_2)$ for all $a_1 \in A_1$, $a_2 \in A_2$.

A GRADED ALGEBRA A over K is a graded module over K together with a pair of degree 0 homomorphisms $\mu: K \to A$ and $\Delta: A \otimes A \to A$ of K-modules such that the diagrams below commute:

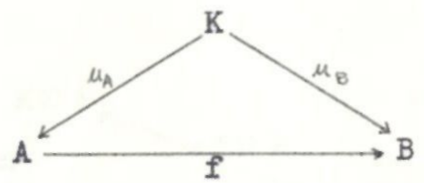


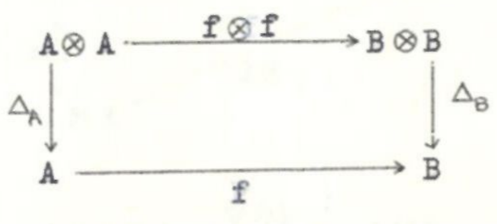


μ is called the UNIT of A and Δ is called the MULTIPLICATION map of A. If $a \in A$ has degree m and $b \in A$ has degree n then ab denotes the element $\Delta(a \otimes b)$ of degree m + n. A graded algebra A over K is said to be AUGMENTED if there exists a degree 0 homomorphism $\epsilon : A \to K$ of K-modules. A is said to be GRADED COMMUTATIVE if $ab = (-1)^{(Deg(a))(Deg(b))}\, ba$ for all $a, b \in A$. A is said to be COMMUTATIVE if ab = ba for all $a, b \in A$.

If A and B are graded algebras over K then $A \otimes B$ is also a graded algebra over K with unit given by the composition $K \xrightarrow{\approx} K \otimes K \xrightarrow{\mu_A \otimes \mu_B} A \otimes B$ and multiplication given by $(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{(Deg(b_1))(Deg(a_2))}(a_1 a_2) \otimes (b_1 b_2)$ for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

If A and B are graded algebras over K then a K-ALGEBRA HOMOMORPHISM or MULTIPLICATIVE map $f : A \to B$ is a K-module homomorphism of degree 0 such that the diagrams below commute:

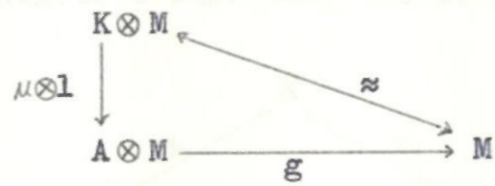


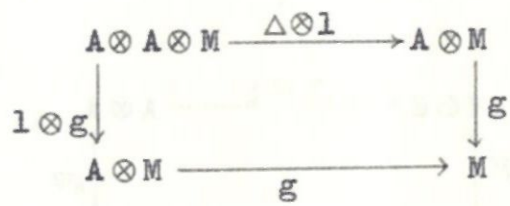


If A_1, A_2, B_1, B_2 are graded algebras over K and $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ are K-algebra homomorphisms, then

$f_1 \otimes f_2 : A_1 \otimes A_2 \to B_1 \otimes B_2$ is a K-algebra homomorphism as well.

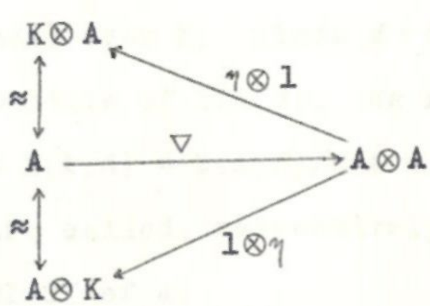
If A is a graded algebra over K then a LEFT A-MODULE M is a graded module over K together with a K-module homomorphism $g: A \otimes M \to M$ of degree 0 such that the diagrams below commute:

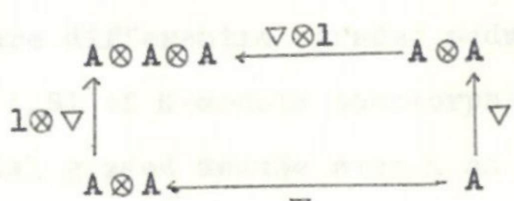




The notion of RIGHT A-MODULE is defined analogously.

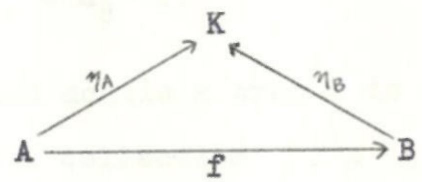
A GRADED COALGEBRA A over K is a graded module over K together with a pair of degree 0 homomorphisms $\eta: A \to K$ and $\nabla: A \to A \otimes A$ of K-modules such that the diagrams below commute:

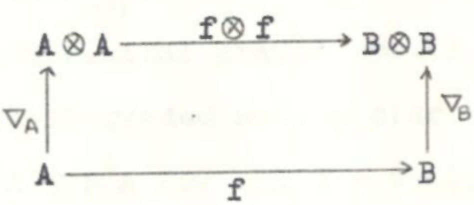




η is called the COUNIT of A and ∇ is called the COMULTIPLICATION map of A.

If A and B are graded coalgebras over K then a K-COALGEBRA HOMOMORPHISM or COMULTIPLICATIVE map $f:A \to B$ is a K-module homomorphism of degree 0 such that the diagrams below commute:





A DIFFERENTIAL graded module A over K is a graded module over K together with a K-module homomorphism $d:A \to A$ of degree +1 such that $d \circ d \equiv 0$. If A is a differential graded module over K, d is called the DIFFERENTIAL. Z(A,d) denotes the graded module Ker(d) over K. B(A,d) denotes the graded module Im(d) over K. Since $d \circ d \equiv 0$, it follows that B(A,d) is a submodule of Z(A,d). We form the quotient graded module $H(A) = H(A,d) = Z(A,d)/B(A,d)$ over K. Z(A,d), B(A,d), and H(A,d) are called, respectively, the CYCLES, BOUNDARIES, and HOMOLOGY of A.

If A and B are differential graded modules over K, then the set Hom(A,B) of K-module homomorphisms from A to B is a differential graded module over K as well; the grading is by homomorphism degree, and the differential is given by $d(f) = d_B \circ f + (-1)^{Deg(f)} f \circ d_A$ for each $f \in Hom(A,B)$. One checks easily that $d \circ d = 0$.

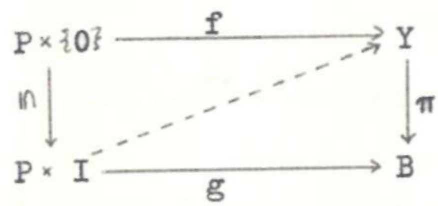
If A and B are differential graded modules over K, then $f \in Hom(A,B)$ is said to be a DIFFERENTIAL K-module homomorphism if $f \circ d_A = d_B \circ f$.

A FILTERED graded module A over K is a graded module over K together with a collection $\{F^iA \mid i = 0,\pm 1,\pm 2,\ldots\}$ of submodules such that $\bigcup_{i \in Z} F^iA = A$ and $F^iA \supseteq F^jA$ whenever $i \leq j$. If A is a differential graded module over K then A is a DIFFERENTIAL FILTERED graded module over K if the filtration also satisfies $d(F^iA) \subseteq F^iA$ for all $i = 0,\pm 1,\pm 2,\ldots$ If A is a graded algebra over K then A is a FILTERED graded ALGEBRA over K if the filtration in this case satisfies $F^iA \cdot F^jA \subseteq F^{i+j}A$ for all $i = 0,\pm 1,\pm 2,\ldots$; $j = 0,\pm 1,\pm 2,\ldots$

For further details on algebraic preliminaries see, for example, Cartan and Eilenberg [7] or MacLane [15].

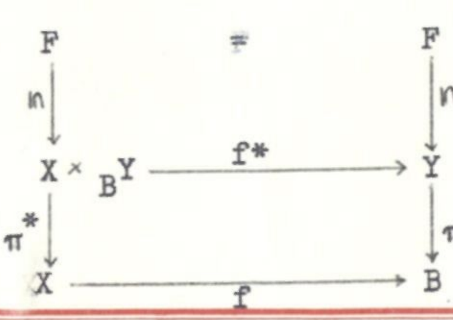
I. 2. GEOMETRY:

A continuous map of topological spaces $\pi: Y \to B$ is said to be a FIBRE MAP if B is path-connected, π is surjective, and any commutative diagram

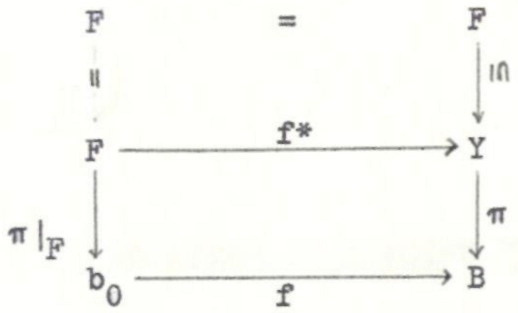


where P is a triangulable space, can be filled in as shown. Fixing a base point $b_0 \in B$ we define $F = \pi^{-1}(b_0)$. F is then unique in the sense that the fibres over any two points have the same singular homology. F is called the FIBRE, Y the FIBRE SPACE, and B the BASE SPACE. The entire collection $F \xrightarrow{\subseteq} Y \xrightarrow{\pi} B$ is called a SERRE FIBRATION.

Given a Serre fibration $F \xrightarrow{\subseteq} Y \xrightarrow{\pi} B$ and a continuous map of topological spaces $f: X \to B$, we form the INDUCED SPACE, denoted $X \times_B Y$, by setting $X \times_B Y = \{(x,y) \in X \times Y \mid f(x) = \pi(y)\}$. In this case there exist natural projections $\pi^*: X \times_B Y \to X$ given by $\pi^*(x,y) = x$ and $f^*: X \times_B Y \to Y$ by $f^*(x,y) = y$ for each $x \in X$ and $y \in Y$. In fact, we have the following commutative diagram



If $X = \{b_0\}$ is a basepoint of B and f is the inclusion map, then $X \times_B Y = \{(x,y) \in X \times Y \mid f(x) = \pi(y)\} =$ $= \{y \in Y \mid b_0 = \pi(y)\} = F$, so in this special case the commutative diagram above becomes



For further details on geometric preliminaries see, for example, Cartan and Eilenberg [7] or MacLane [15].

II. THE COHOMOLOGY OF HOMOGENEOUS SPACES

II. 1. tor, Tor, TOR AND THE TWO-SIDED BAR CONSTRUCTION:

The major theorem in this chapter expresses, under certain reasonable hypotheses, the cohomology of a homogeneous space as a certain torsion product. The proof makes considerable use of various more complicated torsion products, so it is best to describe them first in detail. Each new torsion product will be seen to generalize the previous one. Thus, of course, they also become more and more unwieldy; essentially only the simplest of these products is actually computable. Roughly speaking, our proof will express the cohomology of a homogeneous space first in terms of the middle torsion product, pass to the most complicated, and then in one fell swoop to the simplest, as we desire.

Each torsion product will be defined in terms of some form of the two-sided bar construction. It is worth noting that in each case the two-sided bar construction could also be described by defining the so-called bar construction, tensoring on the two-sides, and noting that the additional structure, namely the differential, is induced naturally from the various components.

The bar construction is due originally to Eilenberg and MacLane [10].

For the remainder of this chapter fix K to be a field. The following material is valid in a somewhat more general context but this will not be needed. A will denote a graded algebra over K with augmentation ϵ. Define $\bar{A} = \mathrm{Ker}(\epsilon)$. Now set $\bar{B}_0(A) = K$, and, for each positive integer n, write

$$\bar{B}_n(A) = \bar{A} \otimes \ldots (n) \ldots \otimes \bar{A}.$$

Finally set

$$\bar{B}(A) = \bigoplus_{n=0}^{\infty} \bar{B}_n(A).$$

An element $[a_1|\ldots|a_n] = [a_1] \otimes \ldots \otimes [a_n] \in \bar{B}(A)$ will have INTERNAL degree $\sum_{i=1}^{n} \mathrm{Deg}(a_i)$, EXTERNAL degree $-n$, bidgree $(\sum_{i=1}^{n} \mathrm{Deg}(a_i), -n)$, and hence degree $\sum_{i=1}^{n} \mathrm{Deg}(a_i) - n$ in the associated graded module over K. $\bar{B}(A)$ is equipped with a natural coproduct

$$\nabla : \bar{B}(A) \to \bar{B}(A) \otimes \bar{B}(A)$$

defined by

$$([a_1|\ldots|a_n]) = \sum_{i=0}^{n} [a_1|\ldots|a_i] \otimes [a_{i+1}|\ldots|a_n],$$

where [ ] will signify $1 \in K$. There is also a natural counit η given by $\eta(k) = k$ for $k \in K \subseteq \bar{B}(A)_0$, and $\eta \equiv 0$ elsewhere. With this structure $\bar{B}(A)$ becomes a graded coalgebra over K.

(a). <u>tor</u>:

Suppose that M is a right A-module and N is a left A-module. We form the complex $M \otimes \bar{B}(A) \otimes N$ with the natural differential d_E given by

$$d_E(m\otimes[a_1|\ldots|a_p]\otimes n) = (-1)^{\mathrm{Deg}(m)} ma_1\otimes[a_2|\ldots|a_p]\otimes n +$$

$$+ \sum_{i=1}^{p-1} (-1)^{\mathrm{Deg}(m) + \sum_{j=1}^{i} \mathrm{Deg}(a_j) - i} m\otimes[a_1|\ldots|a_i a_{i+1}|\ldots|a_p]\otimes n +$$

$$+ (-1)^{\mathrm{Deg}(m) + \sum_{i=1}^{p} \mathrm{Deg}(a_i) - p} m\otimes[a_1|\ldots|a_{p-1}]\otimes a_p n.$$

d_E is called the EXTERNAL differential, since it acts on external degree. We will call the complex

$$(M\otimes \overline{B}(A)\otimes N,\ d_E)$$

the FIRST TWO-SIDED BAR CONSTRUCTION. Observe that the signs have been chosen so that $d_E \circ d_E \equiv 0$. The first two-sided bar construction thus has the structure of a differential graded module over K.

DEFINITION: We define $\mathrm{tor}_A(M,N)$ to be the homology of the first two-sided bar construction:

$$\mathrm{tor}_A(M,N) = H(M\otimes \overline{B}(A)\otimes N,\ d_E).$$

REMARK: It is worth noting that $\mathrm{tor}_A(M,N)$ could be defined in considerably greater generality. Specifically, one could make use of projective resolutions. See Baum [1], for example. Then one would check that the first two-sided bar construction is, in fact, a specific projective resolution.

REMARK: There exist relatively easy ways to compute $\mathrm{tor}_A(M,N)$ in the case where A is a polynomial algebra. See, for example, Baum and Smith [3] for an exposition of the two-sided Koszul construction. We will have more to say about the two-sided Koszul construction in III. In the major theorem in this chapter A will turn out to be a polynomial algebra.

REMARK: Suppose A,M, and N are graded algebras over K and $f:A\to M$ and $g:A\to N$ are multiplicative maps. Then we can regard M as a right A-module by defining a map $M\otimes A\to M$ by $m\otimes a\mapsto mf(a)$, and similarly we can regard N as a left A-module by defining a map $A\otimes N\to N$ by $a\otimes n\mapsto g(a)n$.

(b). Tor:

Now suppose that A is a differential graded algebra over K, M is a right differential A-module, and N is a left diffferential A-module. We again form the complex $M\otimes\overline{B}(A)\otimes N$, this time with the natural differential $d_D =$ $= d_E + d_I$, where

$$d_I(m\otimes[a_1|\dots|a_p]\otimes n) = dm\otimes[a_1|\dots|a_p]\otimes n +$$
$$+ \sum_{i=1}^{p} (-1)^{Deg(m) + \sum_{j=1}^{i-1} Deg(a_j) - (i-1)} m\otimes[a_1|\dots|da_i|\dots|a_p]\otimes n +$$
$$+ (-1)^{Deg(m) + \sum_{i=1}^{p} Deg(a_i) - p} m\otimes[a_1|\dots|a_p]\otimes dn.$$

d_I is called the INTERNAL differential, since it acts on internal degree. We will call the complex

$$(M\otimes\overline{B}(A)\otimes N,\ d_D)$$

the SECOND TWO-SIDED BAR CONSTRUCTION. Observe, again, that the signs have been chosen so that $d_I\circ d_I \equiv 0$, $d_I\circ d_E \equiv -d_E\circ d_I$ and hence that $d_D\circ d_D \equiv 0$. The second two-sided bar construction thus has the structure of a differential graded module over K.

DEFINITION: We define $\mathrm{Tor}_A(M,N)$ to be the homology of the second two-sided bar construction:

$$\mathrm{Tor}_A(M,N) = H(M \otimes \overline{B}(A) \otimes N,\ d_D).$$

REMARK: We note, again, that $\mathrm{Tor}_A(M,N)$ could be defined in considerably greater generality. Specifically, one could make use of differential projective resolutions. See Baum [1][2] or Smith [20], for example. Then one would check that the second two-sided bar construction is, in fact, a specific differential projective resolution. This is essentially done in Smith [20].

REMARK: There exists, again, a description of $\mathrm{Tor}_A(M,N)$, in the case where A is a polynomial algebra, in terms of the two-sided Koszul construction. See Baum and Smith [3] or III.

REMARK: Observe that by assuming the differentials on A, M, and N are zero, d_I disappears; and Tor is therefore a generalization of tor.

REMARK: Suppose A, M, and N are differential graded algebras over K and $f: A \to M$ and $g: A \to N$ are differential multiplicative maps. Then we can regard M as a right differential A-module and N as a left differential A-module in precisely the same way as in (a).

REMARK: The idea of homological algebra with differential operators is due to Borel, H. Cartan, Eilenberg, MacLane, and Moore. The relationship between homological algebra with differential operators and homological algebra without differential operators is the following theorem due to Eilenberg and Moore [11][12]:

<u>THEOREM 1</u>: Let A be a differential graded algebra over K, M be a right differential A-module, and N a left differential A-module. Then there exists a spectral sequence (E_r, d_r), called the EILENBERG - MOORE SPECTRAL SEQUENCE, such that

(i). $E_r \Rightarrow \mathrm{Tor}_A(M,N)$,

(ii). $E_1 = H(M) \otimes \overline{B}(H(A)) \otimes H(N)$ with external differential, i.e., E_1 is the first two-sided bar construction on H(M), H(A), and H(N),

(iii). $E_2 = \mathrm{tor}_{H(A)}(H(M),H(N))$.

<u>REMARK</u>: The Eilenberg - Moore spectral sequence is obtained by filtering $\overline{B}(A)$ on external degree, that is, setting

$$F^q\overline{B}(A) = \{ m \otimes [a_1 | \dots | a_p] \otimes n \mid p \leq q \} .$$

For details on spectral sequences arising from filtrations see Cartan and Eilenberg [7] or MacLane [15]. For a proof of Theorem 1 in a somewhat more general setting see Eilenberg and Moore [11][12], Baum [1], or Smith [20]

<u>THEOREM 2</u>: Consider the following commutative diagram:

$$\begin{array}{ccccc} N_1 & \xleftarrow{\alpha} & A_1 & \xrightarrow{\beta} & M_1 \\ f \uparrow & & \uparrow g & & \uparrow h \\ N_2 & \xleftarrow[\gamma]{} & A_2 & \xrightarrow[\delta]{} & M_2 \end{array}$$

Suppose A_1, A_2, M_1, M_2, N_1, and N_2 are differential graded algebras over K while f, g, h, α, β, γ, and δ are differential multiplicative maps; thus $Tor_{A_1}(M_1,N_1)$, $Tor_{A_2}(M_1,N_1)$, $Tor_{A_2}(M_2,N_2)$, $Tor_{A_2}(M_1,N_2)$, and $Tor_{A_2}(M_2,N_1)$ make sense by a remark above.

(i). The map $\overline{B}Tor_g(1,1)\colon M_1 \otimes \overline{B}(A_2) \otimes N_1 \to M_1 \otimes \overline{B}(A_1) \otimes N_1$ defined by

$$\overline{B}Tor_g(1,1)(m \otimes [a_1|\ldots|a_p] \otimes n) = m \otimes [g(a_1)|\ldots|g(a_p)] \otimes n$$

is a map of differential graded modules and therefore induces a map

$$Tor_g(1,1)\colon Tor_{A_2}(M_1,N_1) \to Tor_{A_1}(M_1,N_1).$$

Furthermore $\overline{B}Tor_g(1,1)$ induces a map $Tor_g(1,1)_r$ of the corresponding spectral sequences such that

$$Tor_g(1,1)_1(m \otimes [a_1|\ldots|a_p] \otimes n) = m \otimes [g_*(a_1)|\ldots|g_*(a_p)] \otimes n.$$

(ii). The map $\overline{B}Tor_1(1,f)\colon M_2 \otimes \overline{B}(A_2) \otimes N_2 \to M_2 \otimes \overline{B}(A_2) \otimes N_1$ defined by

$$\overline{B}Tor_1(1,f)(m \otimes [a_1|\ldots|a_p] \otimes n) = m \otimes [a_1|\ldots|a_p] \otimes f(n)$$

is a map of differential graded modules and therefore induces a map

$$Tor_1(1,f)\colon Tor_{A_2}(M_2,N_2) \to Tor_{A_2}(M_2,N_1).$$

Furthermore $\overline{B}Tor_1(1,f)$ induces a map $Tor_1(1,f)_r$ of the corresponding spectral sequences such that

$$Tor_1(1,f)_1(m \otimes [a_1|\ldots|a_p] \otimes n) = m \otimes [a_1|\ldots|a_p] \otimes f_*(n).$$

(iii). The map $\overline{B}Tor_1(h,1)\colon M_2 \otimes \overline{B}(A_2) \otimes N_2 \to M_1 \otimes \overline{B}(A_2) \otimes N_2$ defined by

$$\overline{B}Tor_1(h,1)(m \otimes [a_1|\ldots|a_p] \otimes n) = h(a) \otimes [a_1|\ldots|a_p] \otimes n$$

is a map of differential graded modules and therefore induces a map

$$Tor_1(h,1)\colon Tor_{A_2}(M_2,N_2) \to Tor_{A_2}(M_1,N_2).$$

Furthermore $\overline{B}Tor_1(h,1)$ induces a map $Tor_1(h,1)_r$ of the corresponding spectral sequences such that

$$Tor_1(h,1)_1(m \otimes [a_1|\ldots|a_p] \otimes n) = h_*(m) \otimes [a_1|\ldots|a_p] \otimes n.$$

<u>REMARK</u>: One way to generalize Theorem 2 would be to consider M_1 and M_2 to be right differential A_1- and A_2-modules, N_1 and N_2 to be left differential A_1- and A_2-modules, respectively, and f and h to be merely differential K-module homomorphisms which are g-semilinear (This means simply that they preserve the appropriate module structure). In this thesis Theorem 2 will be sufficient.

<u>NOTATION</u>: Let $\sigma(i) = (-1)^{Deg(m) + \sum_{j=1}^{i} Deg(a_j) - i}$ for each $i = 0,1,\ldots,p$.

PROOF OF THEOREM 2: The proofs that $\overline{B}Tor_g(1,1)$, $\overline{B}Tor_1(1,f)$ and $\overline{B}Tor_1(h,1)$ are maps of differential graded modules are direct calculations:

(i). $d\overline{B}Tor_g(1,1)(m\otimes[a_1|\ldots|a_p]\otimes n) =$

$d(m\otimes[g(a_1)|\ldots|g(a_p)]\otimes n) = \sigma(0)(m\beta g(a_1)\otimes[g(a_2)|\ldots|g(a_p)]\otimes n) +$

$\sum_{i=1}^{p-1}\sigma(i)(m\otimes[g(a_1)|\ldots|g(a_i)g(a_{i+1})|\ldots|g(a_p)]\otimes n) +$

$\sigma(p)(m\otimes[g(a_1)|\ldots|g(a_{p-1})]\otimes\alpha g(a_p)n) + dm\otimes[g(a_1)|\ldots|g(a_p)]\otimes n +$

$\sum_{i=1}^{p}\sigma(i-1)(m\otimes[g(a_1)|\ldots|dg(a_i)|\ldots|g(a_p)]\otimes n) +$

$\sigma(p)(m\otimes[g(a_1)|\ldots|g(a_p)]\otimes dn) =$

$\sigma(0)(m\beta g(a_1)\otimes[g(a_2)|\ldots|g(a_p)]\otimes n) +$

$\sum_{i=1}^{p-1}\sigma(i)(m\otimes[g(a_1)|\ldots|g(a_ia_{i+1})|\ldots|g(a_p)]\otimes n) +$

$\sigma(p)(m\otimes[g(a_1)|\ldots|g(a_{p-1})]\otimes\alpha g(a_p)n) + dm\otimes[g(a_1)|\ldots|g(a_p)]\otimes n +$

$\sum_{i=1}^{p}\sigma(i-1)(m\otimes[g(a_1)|\ldots|g(da_i)|\ldots|g(a_p)]\otimes n) +$

$\sigma(p)(m\otimes[g(a_1)|\ldots|g(a_p)]\otimes dn) = \overline{B}Tor_g(1,1)d(m\otimes[a_1|\ldots|a_p]\otimes n).$

(ii). $d\overline{B}Tor_1(1,f)(m\otimes[a_1|\ldots|a_p]\otimes n) = d(m\otimes[a_1|\ldots|a_p]\otimes f(n)) =$

$\sigma(0)(m\delta(a_1)\otimes[a_2|\ldots|a_p]\otimes f(n) +$

$\sum_{i=1}^{p-1}\sigma(i)(m\otimes[a_1|\ldots|a_ia_{i+1}|\ldots|a_p]\otimes f(n)) +$

$\sigma(p)(m\otimes[a_1|\ldots|a_{p-1}]\otimes f\gamma(a_p)f(n)) + dm\otimes[a_1|\ldots|a_p]\otimes f(n) +$

$\sum_{i=1}^{p}\sigma(i-1)(m\otimes[a_1|\ldots|da_i|\ldots|a_p]\otimes f(n)) +$

$\sigma(p)(m\otimes[a_1|\ldots|a_p]\otimes df(n)) = \sigma(0)(m\delta(a_1)\otimes[a_2|\ldots|a_p]\otimes f(n) +$

$\sum_{i=1}^{p-1}\sigma(i)(m\otimes[a_1|\ldots|a_ia_{i+1}|\ldots|a_p]\otimes f(n)) +$

$\sigma(p)(m\otimes[a_1|\ldots|a_{p-1}]\otimes f(\gamma(a_p)n)) + dm\otimes[a_1|\ldots|a_p]\otimes f(n) +$

$\sum_{i=1}^{p}\sigma(i-1)(m\otimes[a_1|\ldots|da_i|\ldots|a_p]\otimes f(n)) +$

$\sigma(p)(m\otimes[a_1|\ldots|a_p]\otimes f(dn)) = \overline{B}Tor_1(1,f)d(m\otimes[a_1|\ldots|a_p]\otimes n).$

(iii). The proof of (iii) is completely analogous to the proof of (ii).

The remainder of Theorem 2 now follows immediately.

COROLLARY 3: Under the conditions of Theorem 2...

(i). If $g_*:H(A_2)\rightarrow H(A_1)$ is an isomorphism, then so is

$$Tor_g(1,1):Tor_{A_2}(M_1,N_1)\overset{\approx}{\rightarrow}Tor_{A_1}(M_1,N_1).$$

(ii). If $f_*:H(N_2)\rightarrow H(N_1)$ is an isomorphism, then so is

$$Tor_1(1,f):Tor_{A_2}(M_2,N_2)\overset{\approx}{\rightarrow}Tor_{A_2}(M_2,N_1).$$

(iii). If $h_*:H(M_2)\rightarrow H(N_1)$ is an isomorphism, then so is

$$Tor_1(h,1):Tor_{A_2}(M_2,N_2)\overset{\approx}{\rightarrow}Tor_{A_2}(M_1,N_2).$$

PROOF: In the Eilenberg - Moore spectral sequence the induced maps $Tor_g(1,1)_1$, $Tor_1(1,f)_1$ and $Tor_1(h,1)_1$ are isomorphisms.

DEFINITION: Suppose A and B are differential graded algebras over K and $f,g:A\rightarrow B$ are differential multiplicative maps. We say that f and g are STRONGLY CHAIN HOMOTOPIC

AS MULTIPLICATIVE MAPS if there exists a sequence $\{D^0, D^1, D^2, \ldots\}$ of K-module homomorphisms, with $D^0 : K \to B$ and, for each positive integer n,

$$D^n : A \otimes \ldots (n) \ldots \otimes A \to B,$$

such that

(i). D^n has degree $-n$ for each n.

(ii). D^0 is the identity,

(iii). $dD^n(a_1 \otimes \ldots \otimes a_n) - \sum_{i=1}^{n} \sigma(i-1) D^n(a_1 \otimes \ldots \otimes da_i \otimes \ldots \otimes a_n) =$

$$= \Sigma\, \sigma(i) D^{n-1}(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n) +$$

$$+ \sigma(n) D^{n-1}(a_1 \otimes \ldots \otimes a_{n-1}) f(a_n) - \sigma(1) g(a_1) D^{n-1}(a_2 \otimes \ldots \otimes a_n).$$

WARNING: The summation on the left hand side of (iii) is not $\pm\, D^n d$; the discrepency is due to exterior degree.

THEOREM 4: Suppose A, M, and N are differential graded algebras over K, and $f, g : A \to N$ and $h : A \to M$ are differential multiplicative maps. If f and g are strongly chain homotopic as multiplicative maps, then $\mathrm{Tor}_A(M,N)$ is unambiguously defined; that is, $\mathrm{Tor}_A(M,N)$ is the same whether N is regarded as a left differential A-module via f or via g:

$$(\mathrm{Tor}_A(M,N))_f \approx (\mathrm{Tor}_A(M,N))_g.$$

An analogous result is true for $\mathrm{Tor}_A(N,M)$:

$$(\mathrm{Tor}_A(N,M))_f \approx (\mathrm{Tor}_A(N,M))_g.$$

PROOF: We form $M \otimes \overline{B}(A) \otimes N$ with the differential d_f

obtained via f and with the differential d_g formed via g. Now construct the map

$$\overline{B}D^*: (M\otimes \overline{B}(A)\otimes N,\ d_f)\rightarrow(M\otimes\overline{B}(A)\otimes N,\ d_g)$$

by setting

$$\overline{B}D^*(m\otimes[a_1|\ldots|a_p]\otimes n) = \sum_{i=0}^{p} m\otimes[a_1|\ldots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\ldots\otimes a_p)n.$$

We claim that $\overline{B}D^*$ is a map of differential graded modules; the proof is a direct calculation: $d_g\overline{B}D^*(m\otimes[a_1 \ldots a_p]\otimes n) =$

$$\sum_{i=0}^{p}\sigma(0)(mh(a_1)\otimes[a_2|\ldots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\ldots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i-1}\sigma(j)(m\otimes[a_1|\ldots|a_ja_{j+1}|\ldots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\ldots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sigma(i+1)(m\otimes[a_1|\ldots|a_i]\otimes g(a_{i+1})D^{p-i-1}(a_{i+2}\otimes\ldots\otimes a_p)n) +$$

$$\sum_{i=0}^{p} dm\otimes[a_1|\ldots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\ldots\otimes a_p)n +$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i}\sigma(j-1)(m\otimes[a_1|\ldots|da_j|\ldots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\ldots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sigma(i)(m\otimes[a_1|\ldots|a_i]\otimes dD^{p-i}(a_{i+1}\otimes\ldots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sigma(i)(-1)^{\sum_{j=i+1}^{p}Deg(a_j)-(p-i)}(m\otimes[a_1|\ldots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\ldots\otimes a_p)dn) =$$

$$[\sum_{i=0}^{p}\sigma(0)(mh(a_1)\otimes[a_2|\ldots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\ldots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i-1}\sigma(j)(m\otimes[a_1|\ldots|a_ja_{j+1}|\ldots|a_i]\otimes D^{p-i}(a_{i+2}\otimes\ldots\otimes a_p)n) +$$

$$\sum_{i=0}^{p} dm\otimes[a_1|\ldots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\ldots\otimes a_p)n +$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i}\sigma(j-1)(m\otimes[a_1|\ldots|da_j|\ldots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\ldots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sigma(p)(m\otimes[a_1|\ldots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\ldots\otimes a_p)dn)] +$$

$$[\sum_{i=0}^{p} \sigma(i)(m\otimes[a_1|\dots|a_i]\otimes dD^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p} \sigma(i+1)(m\otimes[a_1|\dots|a_i]\otimes g(a_{i+1})D^{p-i-1}(a_{i+2}\otimes\dots\otimes a_p)n)] =$$

$$[\sum_{i=0}^{p} \sigma(0)(mh(a_1)\otimes[a_2|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i-1} \sigma(j)(m\otimes[a_1|\dots|a_j a_{j+1}\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p} dm\otimes[a_1|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n +$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i} \sigma(j-1)(m\otimes[a_1|\dots|da_j|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p} \sigma(p)(m\otimes[a_1|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)dn)] +$$

$$[\sum_{i=0}^{p}\sum_{j=i+1}^{p}\sigma(i)(-1)^{\sum_{k=i+1}^{j-1}Deg(a_k)-(j-i-1)}(m\otimes[a_1|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes da_j\otimes\dots\otimes a_p)n)$$

$$+ \sum_{i=0}^{p}\sum_{j=i+1}^{p-1}\sigma(j)(m\otimes[a_1|\dots|a_i]\otimes D^{p-i-1}(a_{i+1}\otimes\dots\otimes a_j a_{j+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p} \sigma(p)(m\otimes[a_1|\dots|a_i]\otimes D^{p-i-1}(a_{i+1}\otimes\dots\otimes a_{p-1})f(a_p)n)] =$$

$$\sum_{i=0}^{p} \sigma(0)(mh(a_1)\otimes[a_2|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i-1} \sigma(j)(m\otimes[a_1|\dots|a_j a_{j+1}|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sum_{j=i+1}^{p-1} \sigma(j)(m\otimes[a_1|\dots|a_i]\otimes D^{p-i-1}(a_{i+1}\otimes\dots\otimes a_j a_{j+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p} \sigma(p)(m\otimes[a_1|\dots|a_i]\otimes D^{p-i-1}(a_{i+1}\otimes\dots\otimes a_{p-1})f(a_p)n)] +$$

$$\sum_{i=0}^{p} dm\otimes[a_1|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n +$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i} \sigma(j-1)(m\otimes[a_1|\dots|da_j|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sum_{j=i+1}^{p} \sigma(j-1)(m\otimes[a_1|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes da_j\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p} \sigma(p)(m\otimes[a_1|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)dn) =$$

$= \bar{B}D^* d_f(m \otimes [a_1 | \ldots | a_p] \otimes n)$. Therefore $\bar{B}D^*$ induces a map

$$D^*: (\mathrm{Tor}_A(M,N))_f \to (\mathrm{Tor}_A(M,N))_g.$$

Furthermore $\bar{B}D^*$ induces a map D^*_r of the corresponding spectral sequences such that

$$D^*_1(m \otimes [a_1 | \ldots | a_p] \otimes n) = m \otimes [a_1 | \ldots | a_p] \otimes D^0(1)n =$$

$$= m \otimes [a_1 | \ldots | a_p] \otimes n.$$

Thus D^*_1 is the identity on $E_1 = H(M) \otimes \bar{B}(H(A)) \otimes H(N)$.
The second assertion of Theorem 4 is proved analogously.

<u>REMARK</u>: Observe that D^1 must satisfy, up to signs,

$$dD^1 - D^1 d = f - g.$$

Thus D^1 is a chain homotopy between f and g. It is interesting to note that a stronger condition than just a chain homotopy is necessary to guarantee an isomorphism of Tor here.
In the special case where A is a polynomial algebra, however, a simple chain homotopy is sufficient. The proof of this fact depends on an explicit calculation involving the two-sided Koszul construction, and will be needed in III.
See Baum and Smith [3] or III for details.

(c). <u>TOR</u>:

In Theorems 2 and 4, the comparison theorems for Tor, it is obvious that the fact which makes the proofs go through is the existence of a map of differential graded modules between the appropriate two-sided bar constructions. The following question therefore naturally arises: What are the most general conditions under which there exists such a map?

The answer is precisely the shm theory due to Clark [8], Stasheff [21][22], and Stasheff and Halperin [23]. These papers, particularly Stasheff [21], will serve as basic references for (c). Unfortunately only Clark [8] and Stasheff and Halperin [22] have appeared in print.

NOTATION: For convenience let S(n,k) denote the collection of all k-tuples of positive integers whose sum is n:

$$S(n,k) = \{(i_1,\ldots,i_k) \mid \sum_{j=1}^{k} i_j = n\}.$$

We begin with a result due to Halperin:

THEOREM 5: Suppose A and B are differential graded algebras over K, and $\{f_1, f_2, f_3, \ldots\}$ is a sequence of K-module homomorphisms with

$$f_n\colon A\otimes\ldots(n)\ldots\otimes A\to B$$

for each positive integer n, such that f_n has degree 1-n for each n. Then the map

$$\overline{B}f_*\colon \overline{B}(A)\to\overline{B}(B)$$

defined by

$$\overline{B}f_*([a_1|\ldots|a_n]) = \sum_{k=1}^{n}\sum_{S(n,k)}[f_{i_1}(a_1\otimes\ldots\otimes a_{i_1})|\ldots|f_{i_k}(a_{n-i_k+1}\otimes\ldots\otimes a_n)]$$

is a comultiplicative map.

REMARK: In fact, the above correspondence is one-to-one.

PROOF OF THEOREM 5: We simply observe that

$(\overline{B}f_* \otimes \overline{B}f_*) \nabla([a_1|\ldots|a_n]) =$

$= (\overline{B}f_* \otimes \overline{B}f_*)(\sum_{j=0}^{n} [a_1|\ldots|a_j] \otimes [a_{j+1}|\ldots|a_n]) =$

$= \sum_{j=0}^{n} (\sum_{k=0}^{j} \sum_{S(j,k)} [f_{i_1}(a_1 \otimes \ldots \otimes a_{i_1})|\ldots|f_{i_k}(a_{j-i_k+1} \otimes \ldots \otimes a_j)] \otimes$

$\otimes \sum_{k=0}^{n-j} \sum_{S(n-j,k)} [f_{i_1}(a_{j+1} \otimes \ldots \otimes a_{j+i_1})|\ldots|f_{i_k}(a_{n-i_k+1} \otimes \ldots \otimes a_n)] =$

$= \nabla \overline{B}f_*([a_1|\ldots|a_n]).$

NOTATION: Now regard $\overline{B}(A)$ as a differential graded coalgebra over K by identifying $\overline{B}(A)$ with the canonically isomorphic $K \otimes \overline{B}(A) \otimes K$ and imposing the differential d_D of the second two-sided bar construction.

THEOREM 6: Suppose A and B are differential graded algebras over K, and $\{f_1, f_2, f_3, \ldots\}$ is a sequence of K-module homomorphisms with

$$f_n\colon A \otimes \ldots (n) \ldots \otimes A \to B$$

for each positive integer n, such that

(i). f_n has degree $1 - n$ for each n

(ii). $df_n(a_1 \otimes \ldots \otimes a_n) - \sum_{i=1}^{n} \sigma(i-1) f_n(a_1 \otimes \ldots \otimes da_i \otimes \ldots \otimes a_n) =$

$= \sum_{i=1}^{n-1} \sigma(i)[f_{n-1}(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n)$

$- f_i(a_1 \otimes \ldots \otimes a_i) f_{n-i}(a_{i+1} \otimes \ldots \otimes a_n)].$

Then the map

$$\overline{B}f_*: \overline{B}(A) \to \overline{B}(B)$$

is a differential comultiplicative map.

<u>DEFINITION</u>: A sequence $\{f_1, f_2, f_3, \ldots\}$ satisfying the conditions of Theorem 6 is called a STRONGLY HOMOTOPY MULTIPLICATIVE map (or simply an SHM map) from A to B. Sometimes, by abuse of notation, the first map $f_1: A \to B$ is called shm.

<u>REMARK</u>: In fact the mapping λ from the set of shm maps from A to B into the set of differential comultiplicative maps from $\overline{B}(A)$ to $\overline{B}(B)$ defined by

$$\lambda(\{f_1, f_2, f_3, \ldots\}) = \overline{B}f_*$$

is a one-to-one correspondence.

<u>WARNING</u>: The summation on the left hand side of (ii) in the definition of an shm map is not $f_n d$; again the discrepancy is due to exterior degree.

<u>REMARK</u>: Observe that f_2 must satisfy, up to signs,

$$(df_2 - f_2 d)(a_1 \otimes a_2) = f_1(a_1 a_2) - f_1(a_1) f_1(a_2).$$

Thus f_2 is a chain homotopy measuring how far f_1 deviates from being a multiplicative map. f_3 is a chain homotopy of chain homotopies, and so on.

<u>REMARK</u>: Suppose A and B are differential graded algebras

over K and $f: A \to B$ is a differential multiplicative map. Then the sequence $\{f, 0, 0, 0, \ldots\}$ is clearly an shm map. Unfortunately, even if the first term f_1 of an shm map $\{f_1, f_2, f_3, \ldots\}$ from A to B is a differential multiplicative map, it does not follow that $f_2 = f_3 = f_4 = \ldots = 0$.

PROOF OF THEOREM 6: Pick an arbitrary element $(i_1, \ldots, i_k) \in S(n,k)$ and an arbitrary element $j \in \{1, \ldots, k\}$. Let $\theta(p)$ denote $\sum_{m=1}^{p} i_m$. Now note that

(i). in the expression $d_D \bar{B} f_*([a_1 | \ldots | a_n])$

(a). the term $[f_{i_1}(a_1 \otimes \ldots \otimes a_{\theta(1)}) | \ldots$

$| f_m(a_{\theta(j-1)+1} \otimes \ldots \otimes a_{\theta(j-1)+m}) f_{i_j - m}(a_{\theta(j-1)+m+1} \otimes \ldots \otimes a_{\theta(j)}) | \ldots$

$| f_{i_k}(a_{\theta(k-1)+1} \otimes \ldots \otimes a_n)]$ appears once for each $m \in \{1, \ldots, i_j\}$, with sign $\sigma(\theta(j-1)+m)$, arising from

$(i_1, \ldots, i_{j-1}, m, i_j - m, i_{j+1}, \ldots, i_k) \in S(n, k+1)$;

(b). the term $[f_{i_1}(a_1 \otimes \ldots \otimes a_{\theta(1)}) | \ldots$

$| df_{i_j}(a_{\theta(j-1)+1} \otimes \ldots \otimes a_{\theta(j)}) | \ldots | f_{i_k}(a_{\theta(k-1)+1} \otimes \ldots \otimes a_n)]$

appears once, with sign $\sigma(\theta(j-1))$, arising from $(i_1, \ldots, i_k) \in S(n,k)$;

(ii). in the expression $\bar{B} f_* d_D([a_1 | \ldots | a_n])$

(a). the term $[f_{i_1}(a_1 \otimes \ldots \otimes a_{\theta(1)}) | \ldots$

$| f_{i_j - 1}(a_{\theta(j-1)+1} \otimes \ldots \otimes a_{\theta(j-1)+m} a_{\theta(j-1)+m+1} \otimes \ldots \otimes a_{\theta(j)}) | \ldots$

$| f_{i_k}(a_{\theta(k-1)+1} \otimes \ldots \otimes a_n)]$ appears once for each $m \in \{1, \ldots, i_j - 1\}$, with sign $\sigma(\theta(j-1)+m)$, arising from

$(i_1,\ldots,i_{j-1},i_j-1,i_{j+1},\ldots,i_k) \in S(n-1,k)$;

(b). the term $[f_{i_1}(a_1\otimes\ldots\otimes a_{\theta(1)})|\ldots|$

$|f_{i_j}(a_{\theta(j-1)+1}\otimes\ldots\otimes da_{\theta(j-1)+m}\otimes\ldots\otimes a_{\theta(j)})|\ldots|$

$|f_{i_k}(a_{\theta(k-1)+1}\otimes\ldots\otimes a_n)]$ appears once for each $m \in \{1,\ldots,i_j\}$,

with sign $\sigma(\theta(j-1)+m-1)$, arising from

$(i_1,\ldots,i_{j-1},m,i_j-m,i_{j+1},\ldots,i_k) \in S(n,k+1)$.

Modding out by the sign $\sigma(j-1)$, we notice that (i) and (ii) cancel. Finally, $d_D\overline{B}f_*([a_1|\ldots|a_n])$ is the sum of the terms (i)(a) and (i)(b) over all choices of $(i_1,\ldots,i_k) \in S(n,k)$ and $j \in \{1,\ldots,k\}$, while $\overline{B}f_*([a_1|\ldots|a_n])$ is the sum of the terms (ii)(a) and (ii)(b) over all such choices. This completes the proof.

<u>REMARK</u>: Even normally simple operations tend to be somewhat subtle when dealing with shm maps. For example...

(i). Suppose A,B, and C are differential graded algebras over K, $\{f_1,f_2,f_3,\ldots\}$ is an shm map from A to B, and $\{g_1,g_2,g_3,\ldots\}$ is an shm map from B to C. We must define the composition shm map

$$\{g_1,g_2,g_3,\ldots\}\circ\{f_1,f_2,f_3,\ldots\}$$

from A to C by the rule

$$\lambda^{-1}(\lambda(\{g_1,g_2,g_3,\ldots\})\circ\lambda(\{f_1,f_2,f_3,\ldots\})).$$

There are two important special cases: If $\{f_1, f_2, f_3, \ldots\}$ is an shm map from A to B and g is a differential multiplicative map from B to C, then we have

$$(\{g, 0, 0, 0, \ldots\} \circ \{f_1, f_2, f_3, \ldots\})_n \equiv g \circ f_n.$$

If f is a differential multiplicative map from A to B and $\{g_1, g_2, g_3, \ldots\}$ is an shm map from B to C, then we have

$$(\{g_1, g_2, g_3, \ldots\} \circ \{f, 0, 0, 0, \ldots\})_n = g_n \circ (f \otimes \ldots (n) \ldots \otimes f).$$

(ii). Suppose A,B,C, and D are differential graded algebras over K, $\{f_1, f_2, f_3, \ldots\}$ is an shm map from A to B, and $\{g_1, g_2, g_3, \ldots\}$ is an shm map from C to D. We would like to define the tensor product shm map

$$\{f_1, f_2, f_3, \ldots\} \otimes \{g_1, g_2, g_3, \ldots\}$$

from $A \otimes C$ to $B \otimes D$. We can do this, but there is an unnatural choice to be made. We can define the shm map

$$\{f_1, f_2, f_3, \ldots\} \otimes \{1, 0, 0, 0, \ldots\}$$

from $A \otimes C$ to $B \otimes C$ by setting

$$(\{f_1, f_2, f_3, \ldots\} \otimes \{1, 0, 0, 0, \ldots\})_n((a_1 \otimes c_1) \otimes \ldots \otimes (a_n \otimes c_n)) =$$

$$= f_n(a_1 \otimes \ldots \otimes a_n) \otimes c_1 \ldots c_n.$$

Similarly we can define the shm map

$$\{1, 0, 0, 0, \ldots\} \otimes \{g_1, g_2, g_3, \ldots\}$$

from $B\otimes C$ to $B\otimes D$ by setting

$$(\{1,0,0,0,\ldots\}\otimes\{g_1,g_2,g_3,\ldots\})_n((b_1\otimes c_1)\otimes\ldots\otimes(b_n\otimes c_n)) =$$

$$= b_1\ldots b_n\otimes g_n(c_1\otimes\ldots\otimes c_n).$$

Finally, we can define the shm map

$$\{f_1,f_2,f_3,\ldots\}\otimes\{g_1,g_2,g_3,\ldots\}$$

from $A\otimes C$ to $B\otimes D$ as the composition

$$(\{f_1,f_2,f_3,\ldots\}\otimes\{1,0,0,0,\ldots\})\circ(\{1,0,0,0,\ldots\}\otimes\{g_1,g_2,g_3,\ldots\}).$$

The unnaturality arises because we could just as well define the above tensor product shm map as the compositon

$$(\{1,0,0,0,\ldots\}\otimes\{g_1,g_2,g_3,\ldots\})\circ(\{f_1,f_2,f_3,\ldots\}\otimes\{1,0,0,0,\ldots\}).$$

We do not believe that the two definitions above necessarily coincide. In any case we shall have no occasion to use this general form of the tensor product shm map.

<u>THEOREM 7</u>: If A and B are differential graded algebras over K and $\{f_1,f_2,f_3,\ldots\}$ is an shm map from A to B, then the map $\overline{B}f_*$ induces a map

$$f_*: \mathrm{Tor}_A(K,K)\to\mathrm{Tor}_B(K,K).$$

Furthermore $\overline{B}f_*$ induces a map f_{*r} of the corresponding spectral sequences such that

$$f_{*1}([a_1|\ldots|a_n]) = [f_{1*}(a_1)|\ldots|f_{1*}(a_n)].$$

<u>COROLLARY 8</u>: Under the conditions of Theorem 7, if

$f_{1*}: H(A) \to H(B)$ is an isomorphism, then so is

$$f_*: Tor_A(K,K) \xrightarrow{\approx} Tor_B(K,K).$$

REMARK: Theorem 7 and Corollary 8 may be regarded as a preview of the analogs of Theorems 2 and 4 which are to come. First, however, we must look at more general structures:

DEFINITION: If A is a differential graded algebra over K then a LEFT STRONGLY HOMOTOPY A-MODULE M (or simply a left SH A-module) is a differential graded module over K together with a sequence $\{g_1, g_2, g_3, \ldots\}$ of K-module homomorphisms with

$$g_n: A \otimes \ldots(n)\ldots \otimes A \otimes M \to M$$

for each positive integer n, such that

(i). g_n has degree 1 - n for each n

(ii). $dg_n(a_1 \otimes \ldots \otimes a_n \otimes m)$

$$- \sum_{i=1}^{n} \sigma(i-1) g_n(a_1 \otimes \ldots \otimes da_i \otimes \ldots \otimes a_n \otimes m)$$

$$= \sum_{i=1}^{n-1} \sigma(i) [g_{n-1}(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \otimes m)$$

$$- g_i(a_1 \otimes \ldots \otimes a_i \otimes g_{n-i}(a_{i+1} \otimes \ldots \otimes a_n \otimes m))].$$

The notion of RIGHT STRONGLY HOMOTOPY A-MODULE is defined analagously.

REMARK: Said differently, a sequence $\{g_1, g_2, g_3, \ldots\}$ exhibits M as a left sh A-module provided the sequence

$$\bar{g}_n: A \otimes \ldots(n)\ldots \otimes A \to Hom(M,M)$$

of adjoints is shm.

REMARK: Observe that g_2 must satisfy, up to signs,

$$(dg_2 - g_2 d)(a_1 \otimes a_2 \otimes m) = g_1(a_1 a_2 \otimes n) - g_1(a_1 \otimes g_1(a_2 \otimes n)).$$

Thus g_2 is a chain homotopy measuring how far g_1 deviates from providing the structure of a left differential module. g_3 is a chain homotopy of chain homotopies, and so on.

REMARK: Suppose A, M, and N are differential graded algebras over K, $\{f_1, f_2, f_3, \ldots\}$ is an shm map from A to M, and $\{g_1, g_2, g_3, \ldots\}$ is an shm map from A to N. Then we can regard M as a right sh A-module and N as a left sh A-module by defining the sequence

$$\underline{f}_p : M \otimes A \otimes \ldots (p) \ldots \otimes A \to M$$

by the rule

$$\underline{f}_p(m \otimes a_1 \otimes \ldots \otimes a_p) = mf_p(a_1 \otimes \ldots \otimes a_p),$$

and defining the sequence

$$\underline{g}_p : A \otimes \ldots (p) \ldots \otimes A \otimes N \to N$$

by the rule

$$\underline{g}_p(a_1 \otimes \ldots \otimes a_p \otimes n) = g_p(a_1 \otimes \ldots \otimes a_p)n.$$

CONSTRUCTION: Now suppose that A is a differential graded algebra over K, M is a right sh A-module via the sequence $\{f_1, f_2, f_3, \ldots\}$, and N is a left sh-A-module via the sequence $\{g_1, g_2, g_3, \ldots\}$. We again form the complex

$M \otimes \overline{B}(A) \otimes N$, this time with the natural differential $d_T = d_I + d_S$, where

$$d_S(m \otimes [a_1 | \ldots | a_p] \otimes n) =$$

$$= \sum_{i=1}^{p} \sigma(i+1)(f_i(m \otimes a_1 \otimes \ldots \otimes a_i) \otimes [a_{i+1} | \ldots | a_p] \otimes n) +$$

$$+ \sum_{i=1}^{p-1} \sigma(i)(m \otimes [a_1 | \ldots a_i a_{i+1} | \ldots a_p] \otimes n) +$$

$$+ \sum_{i=0}^{p-1} \sigma(i+1)(m \otimes [a_1 | \ldots | a_i] \otimes g_{p-i}(a_{i+1} \otimes \ldots \otimes a_p \otimes n)).$$

d_S is called the STRONGLY HOMOTOPY MODULAR (or simply the SHM) differential. We will call the complex

$$(M \otimes \overline{B}(A) \otimes N,\ d_T)$$

the THIRD TWO-SIDED BAR CONSTRUCTION. Observe, again, that the signs have been chosen so that $d_S \circ d_S + d_S \circ d_I + d_I \circ d_S \equiv 0$ and hence that $d_T \circ d_T \equiv 0$. The third two-sided bar construction thus has the structure of a differential graded module over K.

<u>DEFINITION</u>: We define $TOR_A(M,N)$ to be the homology of the third two-sided bar construction:

$$TOR_A(M,N) = H(M \otimes \overline{B}(A) \otimes N,\ d_T).$$

<u>REMARK</u>: Although we believe it is possible, we do not yet know how to define $TOR_A(M,N)$ in greater generality. In particular, it would be extremely useful to be able to describe $TOR_A(M,N)$ in the case where A is a polynomial algebra in terms of some form of two-sided Koszul construction, as results in III will suggest.

REMARK: Observe that if M is a right differential A-module via a map $f: M \otimes A \to M$ and N is a left differential A-module via a map $g: A \otimes N \to N$, then the sequence $\{f, 0, 0, 0, \ldots\}$ exhibits M as a right sh A-module, the sequence $\{g, 0, 0, 0, \ldots\}$ exhibits N as a left sh A-module, and we have $d_S \equiv d_E$; thus TOR is a generalization of Tor.

REMARK: Again filtering $\overline{B}(A)$ on external degree, the proof of the following analog of Theorem 1 goes over essentially word for word:

THEOREM 9: Let A be a differential graded algebra over K, M be a right sh A-module, and N be a left sh-A-module. Then there exists a spectral sequence (E_r, d_r), which we will also call the EILENBERG - MOORE SPECTRAL SEQUENCE, such that

(i). $E_r \Longrightarrow TOR_A(M,N)$,

(ii). $E_1 = H(M) \otimes \overline{B}(H(A)) \otimes H(N)$ with external differential, i.e., E_1 is the first two-sided bar construction on H(M), H(A), and H(N).

(iii). $E_2 = tor_{H(A)}(H(M), H(N))$.

THEOREM 10: Consider the following commutative diagrams:

$$\begin{array}{ccccc}
N_1 & \xleftarrow{\ \alpha\ } & A_1 & \xrightarrow{\ \beta\ } & M_2 \\
\uparrow{\scriptstyle f_n} & & \uparrow{\scriptstyle g_n} & & \uparrow{\scriptstyle h_n} \\
N_2 \otimes \ldots (n) \ldots \otimes N_2 & \xleftarrow{\gamma \otimes \ldots (n) \ldots \otimes \gamma} & A_2 \otimes \ldots (n) \ldots \otimes A_2 & \xrightarrow{\delta \otimes \ldots (n) \ldots \otimes \delta} & M_2 \otimes \ldots (n) \ldots \otimes M_2
\end{array}$$

Suppose A_1, A_2, M_1, M_2, N_1, and N_2 are differential graded algebras over K while $\{f_1, f_2, f_3, \ldots\}$, $\{g_1, g_2, g_3, \ldots\}$, and $\{h_1, h_2, h_3, \ldots\}$ are shm maps, and α, β, γ, and δ are differential multiplicative maps; thus $Tor_{A_1}(M_1, N_1)$, $TOR_{A_2}(M_1, N_1)$, $Tor_{A_2}(M_2, N_2)$, $TOR_{A_2}(M_1, N_2)$ and $TOR_{A_2}(M_2, N_1)$ make sense by remarks above.

(i). The map $\overline{B}TOR_{g_*}(1,1): M_1 \otimes \overline{B}(A_2) \otimes N_1 \to M_1 \otimes \overline{B}(A_1) \otimes N_1$ defined by

$$\overline{B}TOR_{g_*}(1,1)(m \otimes [a_1 | \ldots | a_p] \otimes n) = m \otimes \overline{B}g_*([a_1 | \ldots | a_p]) \otimes n$$

is a map of differential graded modules and therefore induces a map

$$TOR_{g_*}(1,1): TOR_{A_2}(M_1, N_1) \to Tor_{A_1}(M_1, N_1).$$

Furthermore $\overline{B}TOR_{g_*}(1,1)$ induces a map $TOR_{g_*}(1,1)_r$ of the corresponding spectral sequences such that

$$TOR_{g_*}(1,1)_1(m \otimes [a_1 | \ldots | a_p] \otimes n) = m \otimes [g_{1*}(a_1) | \ldots | g_{1*}(a_p)] \otimes n.$$

(ii). The map $\overline{B}TOR_1(1, f_*): M_2 \otimes \overline{B}(A_2) \otimes N_2 \to M_2 \otimes \overline{B}(A_1) \otimes N_1$ defined by

$$\overline{B}TOR_1(1, f_*)(m \otimes [a_1 | \ldots | a_p] \otimes n) =$$

$$\sum_{i=0}^{p} m \otimes [a_1 | \ldots | a_i] \otimes f_{p-i+1}(\gamma(a_{i+1}) \otimes \ldots \otimes \gamma(a_p) \otimes n)$$

is a map of differential graded modules and therefore induces a map

$$TOR_1(1, f_*): Tor_{A_2}(M_2, N_2) \to TOR_{A_2}(M_2, N_1).$$

Furthermore $\overline{B}TOR_1(1,f_*)$ induces a map $TOR_1(1,f_*)$ of the corresponding spectral sequences such that

$$TOR_1(1,f_*)_1(m\otimes[a_1|\dots|a_p]\otimes n) = m\otimes[a_1|\dots|a_p]\otimes f_{1*}(n).$$

(iii). The map $\overline{B}TOR_1(h_*,1)\colon M_2\otimes\overline{B}(A_2)\otimes N_2\to M_1\otimes\overline{B}(A_2)\otimes N_2$ defined by

$$\overline{B}TOR_1(h_*,1)(m\otimes[a_1|\dots|a_p]\otimes n) =$$

$$\sum_{i=0}^{p} h_{i+1}(m\otimes\delta(a_1)\otimes\dots\otimes\delta(a_i))\otimes[a_{i+1}|\dots|a_p]\otimes n$$

is a map of differential graded modules and therefore induces a map

$$TOR_1(h_*,1)\colon Tor_{A_2}(M_2,N_2)\to TOR_{A_2}(M_1,N_2).$$

Furthermore $\overline{B}TOR_1(h_*,1)$ induces a map $TOR_1(h_*,1)$ of the corresponding spectral sequences such that

$$TOR_1(h_*,1)_1(m\otimes[a_1|\dots|a_p]\otimes n) = h_{1*}(m)\otimes[a_1|\dots|a_p]\otimes n.$$

<u>PROOF</u>: The proofs that $\overline{B}TOR_{g_*}(1,1)$, $\overline{B}TOR_1(1,f_*)$, and $\overline{B}TOR_1(h_*,1)$ are maps of differential graded modules are direct calculations:

(i). $d_D\overline{B}TOR_{g_*}(1,1)(m\otimes[a_1|\dots|a_p]\otimes n) =$

$d_D(m\otimes\overline{B}g_*([a_1|\dots|a_p])\otimes n) =$

$\sum_{i=1}^{p}\sigma(0)(m\beta(g_i(a_1\otimes\dots\otimes a_i))\otimes\overline{B}g_*([a_{i+1}|\dots|a_p])\otimes n) +$

$\sum_{i=0}^{p-1}\sigma(p)(m\otimes\overline{B}g_*([a_1|\dots|a_i])\otimes\alpha(g_{p-i}(a_{i+1}\otimes\dots\otimes a_p))n) +$

$$dm\otimes\bar{B}g_*([a_1|\ldots|a_p])\otimes n + \sigma(p)(m\otimes\bar{B}g_*([a_1|\ldots|a_p])\otimes dn) +$$

$$\sigma(0)(m\otimes d_D\bar{B}g_*([a_1|\ldots|a_p])\otimes n) =$$

$$\sum_{i=1}^{p}\sigma(0)(m\beta(g_i(a_1\otimes\ldots\otimes a_i))\otimes\bar{B}g_*([a_{i+1}|\ldots|a_p])\otimes n) +$$

$$\sum_{i=0}^{p-1}\sigma(p)(m\otimes\bar{B}g_*([a_1|\ldots|a_i])\otimes\alpha(g_{p-i}(a_{i+1}\otimes\ldots\otimes a_p))n) +$$

$$dm\otimes\bar{B}g_*([a_1|\ldots|a_p])\otimes n + \sigma(p)(m\otimes\bar{B}g_*([a_1|\ldots|a_p])\otimes dn) +$$

$$\sigma(0)(m\otimes\bar{B}g_*d_D([a_1|\ldots|a_p])\otimes n) = \overline{B}\mathrm{TOR}_{g_*}(1,1)d_T(m\otimes[a_1|\ldots|a_p]\otimes n).$$

(ii). $d_T\overline{B}\mathrm{TOR}_1(1,f_*)(m\otimes[a_1|\ldots|a_p]\otimes n) =$

$$\sum_{i=0}^{p}d_T(m\otimes[a_1|\ldots|a_i]\otimes f_{p-i+1}(\gamma(a_{i+1})\otimes\ldots\otimes\gamma(a_p)\otimes n)) =$$

$$\sum_{i=0}^{p}dm\otimes[a_1|\ldots|a_i]\otimes f_{p-i+1}(\gamma(a_{i+1})\otimes\ldots\otimes\gamma(a_p)\otimes n) +$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i}\sigma(j-1)(m\otimes[a_1|\ldots|da_j|\ldots|a_i]\otimes f_{p-i+1}(\gamma(a_{i+1})\otimes\ldots\otimes\gamma(a_p)\otimes n)) +$$

$$\sum_{i=0}^{p}\sigma(i)(m\otimes[a_1|\ldots|a_i]\otimes df_{p-i+1}(\gamma(a_{i+1})\otimes\ldots\otimes\gamma(a_p)\otimes n)) +$$

$$\sum_{i=0}^{p}\sigma(0)(m\delta(a_1)\otimes[a_2|\ldots|a_i]\otimes f_{p-i+1}(\gamma(a_{i+1})\otimes\ldots\otimes\gamma(a_p)\otimes n)) +$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i-1}\sigma(j)(m\otimes[a_1|\ldots|a_ja_{j+1}|\ldots|a_i]\otimes f_{p-i+1}(\gamma(a_{i+1})\otimes\ldots\otimes\gamma(a_p)\otimes n)) +$$

$$\sum_{i=0}^{p}\sum_{j=i+1}^{p}\sigma(j)(m\otimes[a_1|\ldots|a_i]\otimes f_{j-i}(\gamma(a_{i+1})\otimes\ldots\otimes\gamma(a_j))f_{p-j+1}(\gamma(a_{j+1})\otimes$$

$$\ldots\otimes\gamma(a_p)\otimes n)) = \sum_{i=0}^{p}dm\otimes[a_1|\ldots|a_i]\otimes f_{p-i+1}(\gamma(a_{i+1})\otimes\ldots\otimes\gamma(a_p)\otimes n) +$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i}\sigma(j-1)(m\otimes[a_1|\ldots|da_j|\ldots|a_i]\otimes f_{p-i+1}(\gamma(a_{i+1})\otimes\ldots\otimes\gamma(a_p)\otimes n)) +$$

$$\sum_{i=0}^{p}\sum_{j=i+1}^{p}\sigma(j-1)(m\otimes[a_1|\ldots|a_i]\otimes f_{p-i+1}(\gamma(a_{i+1})\otimes\ldots\otimes\gamma(da_j)\otimes\ldots\otimes$$

$$\gamma(a_p)\otimes n)) + \sum_{i=0}^{p}\sigma(p)(m\otimes[a_1|\ldots|a_i]\otimes f_{p-i+1}(\gamma(a_{i+1})\otimes\ldots\otimes\gamma(a_p)\otimes dn)) +$$

$\sum_{i=0}^{p} \sigma(0)(m\delta(a_1)\otimes[a_2|\ldots|a_i]\otimes f_{p-i+1}(\gamma(a_{i+1})\otimes\ldots\otimes\gamma(a_p)\otimes n))$ +

$\sum_{i=0}^{p}\sum_{j=1}^{i-1} \sigma(j)(m\otimes[a_1|\ldots|a_j a_{j+1}|\ldots|a_i]\otimes f_{p-i+1}(\gamma(a_{i+1})\otimes\ldots\otimes\gamma(a_p)\otimes n))$ +

$\sum_{i=0}^{p}\sum_{j=i+1}^{p-1} \sigma(j)(m\otimes[a_1|\ldots|a_i]\otimes f_{p-i}(\gamma(a_{i+1})\otimes\ldots\otimes\gamma(a_j a_{j+1})\otimes\ldots\otimes\gamma(a_p)\otimes n))$

$+ \sum_{i=0}^{p}\sigma(p)(m\otimes[a_1|\ldots|a_i]\otimes f_{p-i}(\gamma(a_{i+1})\otimes\ldots\otimes\gamma(a_{p-1})\otimes\gamma(a_p)n))$ =

$\overline{B}TOR_1(1,f_*)d_D(m\otimes[a_1|\ldots|a_p]\otimes n)$.

(iii). The proof of (iii) is completely analogous to the proof of (ii).

The remainder of Theorem 10 now follows immediately.

<u>COROLLARY 11</u>: Under the conditions of Theorem 10...

(i). If $g_{1*}:H(A_2)\to H(A_1)$ is an isomorphism, then so is

$$TOR_{g_*}(1,1):TOR_{A_2}(M_1,N_1)\xrightarrow{\approx}TOR_{A_1}(M_1,N_1).$$

(ii). If $f_{1*}:H(N_2)\to H(N_1)$ is an isomorphism, then so is

$$TOR_1(1,f_*):Tor_{A_2}(M_2,N_2)\xrightarrow{\approx}TOR_{A_2}(M_2,N_1).$$

(iii). If $h_{1*}:H(M_2)\to H(M_1)$ is an isomorphism, then so is

$$TOR_1(h_*,1):Tor_{A_2}(M_2,N_2)\xrightarrow{\approx}TOR_{A_2}(M_1,N_2).$$

<u>PROOF</u>: In the Eilenberg - Moore spectral sequence the induced maps $TOR_{g_*}(1,1)_1$, $TOR_1(1,f_*)_1$, and $TOR_1(h_*,1)_1$ are isomorphisms.

<u>DEFINITION</u>: Suppose A and B are differential graded algebras over K and $\{f_1,f_2,f_3,\ldots\}$ and $\{g_1,g_2,g_3,\ldots\}$ are shm maps from A to B. We say that $\{f_1,f_2,f_3,\ldots\}$

and $\{g_1, g_2, g_3, \ldots\}$ are STRONGLY CHAIN HOMOTOPIC AS SHM MAPS if there exists a sequence $\{D^0, D^1, D^2, \ldots\}$ of K-module homomorphisms with $D^0 : K \to B$ and, for each positive integer n,

$$D^n : A \otimes \ldots (n) \ldots \otimes A \to B,$$

such that

(i). D^n has degree $-n$ for each n.

(ii). D^0 is the identity.

(iii). $dD^n(a_1 \otimes \ldots \otimes a_n) - \sum_{i=1}^{n} \sigma(i-1) D^n(a_1 \otimes \ldots \otimes da_i \otimes \ldots \otimes a_n) =$

$$= \sum_{i=1}^{n-1} \sigma(i) D^{n-1}(a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n) +$$

$$+ \sum_{i=1}^{n-1} \sigma(n1) D^i(a_1 \otimes \ldots \otimes a_i) f_{n-i}(a_{i+1} \otimes \ldots \otimes a_n)$$

$$- \sum_{i=1}^{n-1} \sigma(i) g_i(a_1 \otimes \ldots \otimes a_i) D^{n-i}(a_{i+1} \otimes \ldots \otimes a_n).$$

<u>REMARK</u>: Observe, of course, that the notion of strongly chain homotopic shm maps generalizes the notion of strongly chain homotopic multiplicative maps in the usual way.

<u>THEOREM 12</u>: Suppose A, M, and N are differential graded algebras over K, and $\{f_1, f_2, f_3, \ldots\}$ and $\{g_1, g_2, g_3, \ldots\}$ are shm maps from A to N, while $\{h_1, h_2, h_3, \ldots\}$ is an shm map from A to M. If $\{f_1, f_2, f_3, \ldots\}$ and $\{g_1, g_2, g_3, \ldots\}$ are strongly chain homotopic as shm maps, then $TOR_A(M,N)$ is unambiguously defined; that is, $TOR_A(M,N)$ is the same whether N is regarded as a left sh A-module via f or via g:

$$(TOR_A(M,N))_{f_*} \approx (TOR_A(M,N))_{g_*}.$$

An analogous result is true for $TOR_A(N,M)$:

$$(\mathrm{TOR}_A(N,M))_{f_*} \approx (\mathrm{TOR}_A(N,M))_{g_*}.$$

PROOF: We form $M\otimes\overline{B}(A)\otimes N$ with the differential d_{f_*} obtained via $\{f_1,f_2,f_3,\dots\}$ and with the differential d_{g_*} obtained via $\{g_1,g_2,g_3,\dots\}$. Now construct the map

$$\overline{B}D^*:\ (M\otimes\overline{B}(A)\otimes N,\ d_{f_*})\rightarrow(M\otimes\overline{B}(A)\otimes N,\ d_{g_*})$$

by setting

$$\overline{B}D^*(m\otimes[a_1|\dots|a_p]\otimes n) = \sum_{i=0}^{p} m\otimes[a_1|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n.$$

We claim that $\overline{B}D^*$ is a map of differential graded modules; the proof is a direct calculation: $d_{g_*}\overline{B}D^*(m\otimes[a_1|\dots|a_p]\otimes n) =$

$\sum_{i=0}^{p}\sum_{j=1}^{i}\sigma(0)(mh_j(a_1\otimes\dots\otimes a_j)\otimes[a_{j+1}|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$

$\sum_{i=0}^{p}\sum_{j=1}^{i-1}\sigma(j)(m\otimes[a_1|\dots|a_ja_{j+1}|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$

$\sum_{i=0}^{p}\sum_{j=i+1}^{p}\sigma(j)(m\otimes[a_1|\dots|a_i]\otimes g_{j-i}(a_{i+1}\otimes\dots\otimes a_j)D^{p-j}(a_{j+1}\otimes\dots\otimes a_p)n) +$

$\sum_{i=0}^{p} dm\otimes[a_1|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n +$

$\sum_{i=0}^{p}\sum_{j=1}^{i}\sigma(j-1)(m\otimes[a_1|\dots|da_j|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$

$\sum_{i=0}^{p}\sigma(i)(m\otimes[a_1|\dots|a_i]\otimes dD^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$

$\sum_{i=0}^{p}\sigma(p)(m\otimes[a_1|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)dn) =$

$[\sum_{i=0}^{p}\sum_{j=1}^{i}\sigma(0)(mh_j(a_1\otimes\dots\otimes a_j)\otimes[a_{j+1}|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$

$\sum_{i=0}^{p}\sum_{j=1}^{i-1}\sigma(j)(m\otimes[a_1|\dots|a_ja_{j+1}|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$

$\sum_{i=0}^{p} dm\otimes[a_1|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n +$

$$\sum_{i=0}^{p}\sum_{j=1}^{i} \sigma(j-1)(m\otimes[a_1|\dots|da_j|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p} \sigma(p)(m\otimes[a_1|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)dn)] +$$

$$\sum_{i=0}^{p} \sigma(i)(m\otimes[a_1|\dots|a_i]\otimes dD^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sum_{j=i+1}^{p} \sigma(j)(m\otimes[a_1|\dots|a_i]\otimes g_{j-i}(a_{i+1}\otimes\dots\otimes a_j)D^{p-j}(a_{j+1}\otimes\dots\otimes a_p)n)] =$$

$$[\sum_{i=0}^{p}\sum_{j=0}^{i} \sigma(0)(mh_j(a_1\otimes\dots\otimes a_j)\otimes[a_{j+1}|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i-1} \sigma(j)(m\otimes[a_1|\dots|a_ja_{j+1}|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p} dm\otimes[a_1|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i} \sigma(j-1)(m\otimes[a_1|\dots|da_j|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p} \sigma(p)(m\otimes[a_1|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)dn)] +$$

$$[\sum_{i=0}^{p}\sum_{j=i+1}^{p} \sigma(j-1)(m\otimes[a_1|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes da_j\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sum_{j=i+1}^{p-1} \sigma(j)(m\otimes[a_1|\dots|a_i]\otimes D^{p-i-1}(a_{i+1}\otimes\dots\otimes a_ja_{j+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sum_{j=i}^{p} \sigma(p)(m\otimes[a_1|\dots|a_i]\otimes D^{j-i}(a_{i+1}\otimes\dots\otimes a_j)f_{p-j}(a_{i+1}\otimes\dots\otimes a_p)n)] =$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i} \sigma(0)(mh_j(a_1\otimes\dots\otimes a_j)\otimes[a_{j+1}|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i-1} \sigma(j)(m\otimes[a_1|\dots|a_ja_{j+1}|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sum_{j=i+1}^{p-1} \sigma(j)(m\otimes[a_1|\dots|a_i]\otimes D^{p-i-1}(a_{i+1}\otimes\dots\otimes a_ja_{j+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p}\sum_{j=i}^{p} \sigma(p)(m\otimes[a_1|\dots|a_i]\otimes D^{j-i}(a_{i+1}\otimes\dots\otimes a_j)f_{p-j}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$$\sum_{i=0}^{p} dm\otimes[a_1|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n +$$

$$\sum_{i=0}^{p}\sum_{j=1}^{i} \sigma(j-1)(m\otimes[a_1|\dots|da_j|\dots|a_i]\otimes D^{p-i}(a_{i+1}\otimes\dots\otimes a_p)n) +$$

$\sum_{i=0}^{p} \sum_{j=i+1}^{p} \sigma(j-1)(m \otimes [a_1 | \ldots | a_i] \otimes D^{p-i}(a_{i+1} \otimes \ldots \otimes da_j \otimes \ldots \otimes a_p)n) +$

$\sum_{i=0}^{p} \sigma(p)(m \otimes [a_1 | \ldots | a_i] \otimes D^{p-i}(a_{i+1} \otimes \ldots \otimes a_p)dn) =$

$\overline{B}D^* d_{f_*}(m \otimes [a_1 | \ldots | a_p] \otimes n)$. Therefore $\overline{B}D^*$ induces a map

$$D^*: (TOR_A(M,N))_{f_*} \longrightarrow (TOR_A(M,N))_{g_*}.$$

Furthermore $\overline{B}D^*$ induces a map D^*_r of the corresponding spectral sequences such that

$$D^*_1(m \otimes [a_1 | \ldots | a_p] \otimes n) = m \otimes [a_1 | \ldots | a_p] \otimes D^0(1)n =$$

$$= m \otimes [a_1 | \ldots | a_p] \otimes n.$$

Thus D^*_1 is the identity on $E_1 = H(M) \otimes \overline{B}(H(A)) \otimes H(N)$. The second assertion of Theorem 12 is proved analogously.

II. 2. THE SHM MAP FROM H*(BG;K) INTO C*(BG;K):

For the purposes of the main theorem of this chapter we shall require our shm map $\{\theta_1, \theta_2, \theta_3, \ldots\}$ from H*(BG;K) into C*(BG;K) to have certain very desirable properties. However, to begin with we shall simply sketch an existence proof for an shm map $\{\phi_1, \phi_2, \phi_3, \ldots\}$ without any particular special properties.

DEFINITION: Let A be a differential graded algebra over K and Δ denote its multiplication. A is said to be STRONGLY HOMOTOPY COMMUTATIVE (or simply SHC) if $\Delta = \Delta_1$ is the first term of a sequence $\{\Delta_1, \Delta_2, \Delta_3, \ldots\}$ of K-module homomorphisms with

$$\Delta_n : A \otimes \ldots (2n) \ldots \otimes A \to A$$

for each positive integer n, such that

(i). Δ_n has degree 1-n for each n.

(ii). If we denote by c_i the element $a_i \otimes b_i \in A \otimes A$, for each i = 1,...,n, then $(d\Delta_n + \Delta_n d)(c_1 \otimes \ldots \otimes c_n) =$

$$= \sum_{i=1}^{n-1} (-1)^i (\Delta_{n-1}(c_1 \otimes \ldots \otimes c_i c_{i+1} \otimes \ldots \otimes c_n)$$

$$- \Delta_i(c_1 \otimes \ldots \otimes c_i) \Delta_{n-i}(c_{i+1} \otimes \ldots \otimes c_n)).$$

WARNING: We are nearly saying that $\Delta = \Delta_1$ is the first term of an shm map $\{\Delta_1, \Delta_2, \Delta_3, \ldots\}$ from $A \otimes A$ into A, but not quite; the discrepency is, as usual, in the signs.

REMARK: Consider the cochains C*(X;K) of a topological space. Recall that $C^*(X;K)$ is a differential graded algebra over K with multiplication given by $\cup$-product

$$\cup : C^*(X;K) \otimes C^*(X;K) \to C^*(X;K).$$

Recall also that there exists a $\cup_1$-product

$$\cup_1 : C^*(X;K) \otimes C^*(X;K) \to C^*(X;K)$$

which makes our $\cup$-product homotopy commutative, i.e.,

(i). $\cup_1$ has degree -1.

(ii). $d(a \cup_1 b) = ab - (-1)^{Deg(a)Deg(b)} ba - da \cup_1 b$
$- (-1)^{Deg(a)} a \cup_1 db.$

(iii). $(ab \cup_1 c) = (-1)^{Deg(a)} a(b \cup_1 c) + (-1)^{Deg(b)Deg(c)} (a \cup_1 c)b$

(iii) is known as the HIRSCH FORMULA. It is interesting to note that no such anaolog of (iii) exists for $(a \cup_1 bc)$. The following theorem is based on work of Dold [9]:

THEOREM 1: If X is a topological space then $C^*(X;K)$ is shc.

SKETCH OF PROOF: We agree that $\triangle_1 = \cup : C^*(X;K) \otimes C^*(X;K) \longrightarrow C^*(X;K)$ shall be the first map of the sequence. Construct

$$\triangle_2 : C^*(X;K) \otimes \ldots (4) \ldots \otimes C^*(X;K) \to C^*(X;K)$$

by setting

$$\triangle_2(a_1 \otimes b_1 \otimes a_2 \otimes b_2) = (-1)^{Deg(a_1)} a_1(b_1 \cup_1 a_2) b_2.$$

This is justified because $(d\triangle_2 + \triangle_2 d)(c_1 \otimes c_2) =$

$$= (-1)^{Deg(a_1)}da_1(b_1 \cup_1 a_2)b_2 + (-1)^{2Deg(a_1)}a_1b_1a_2b_2 +$$

$$- (-1)^{2Deg(a_1)+Deg(b_1)Deg(a_2)}a_1a_2b_1b_2 - (-1)^{2Deg(a_1)}a_1(db_1 \cup_1 a_2)b_2$$

$$- (-1)^{2Deg(a_1)+Deg(b_1)}a_1(b_1 \cup_1 da_2)b_2 +$$

$$(-1)^{2Deg(a_1)+Deg(a_2)+Deg(b_1)-1}a_1(b_1 \cup_1 a_2)db_2 +$$

$$(-1)^{Deg(a_1)+1}da_1(b_1 \cup_1 a_2)b_2 + (-1)^{2Deg(a_1)}a_1(db_1 \cup_1 a_2)b_2 +$$

$$(-1)^{2Deg(a_1)+Deg(b_1)}a_1(b_1 \cup_1 da_2)b_2 +$$

$$(-1)^{2Deg(a_1)+Deg(a_2)+Deg(b_1)}a_1(b_1 \cup_1 a_2)db_2 = a_1b_1a_2b_2$$

$$- (-1)^{Deg(a_2)+Deg(b_1)}a_1a_2b_1b_2 = \Delta_1(c_1)\Delta_1(c_2) - \Delta_1(c_1c_2).$$

Now the main theorem in Dold [9] may be stated:

$$\mathrm{Hom}(C^*(X;K)\otimes \ldots(n)\ldots \otimes C^*(X;K),\ C^*(X\times\ldots(n)\ldots\times X;K))$$

is acyclic. Using this fact and the strict symmetry of the topological diagonal, we construct the higher maps $\Delta_3, \Delta_4, \Delta_5, \ldots$:

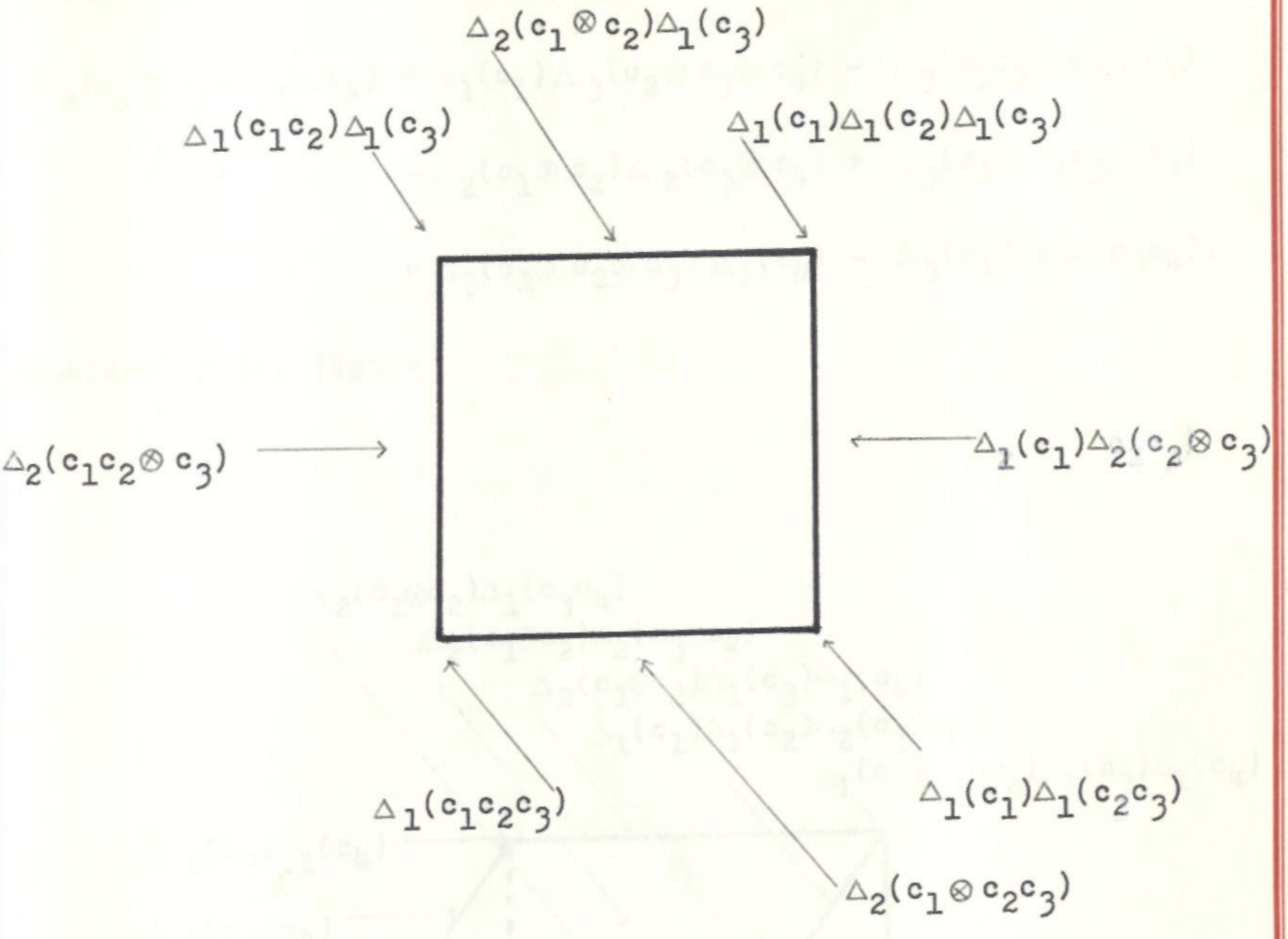
To construct the map Δ_3, consider the map

$$\nabla_3\colon C^*(X;K)\otimes \ldots(6)\ldots \otimes C^*(X;K) \to C^*(X;K)$$

defined by

$$\nabla_3(c_1\otimes c_2\otimes c_3) = \Delta_1(c_1)\Delta_2(c_2\otimes c_3) - \Delta_2(c_1c_2\otimes c_3)$$
$$- \Delta_2(c_1\otimes c_2)\Delta_1(c_3) + \Delta_2(c_1\otimes c_2c_3).$$

Considering the figure



We thus observe that ∇_3 is a cycle; hence ∇_3 is a boundary: that is, there exists

$$\Delta_3 : C^*(X;K) \otimes \ldots (6) \ldots \otimes C^*(X;K) \to C^*(X;K)$$

such that

(i). Δ_3 has degree -2

(ii). $(d\Delta_3 + \Delta_3 d)(c_1 \otimes c_2 \otimes c_3) = \nabla_3(c_1 \otimes c_2 \otimes c_3)$, as desired.

To construct the map Δ_4, consider the map

$$\nabla_4 : C^*(X;K) \otimes \ldots (8) \ldots \otimes C^*(X;K) \to C^*(X;K)$$

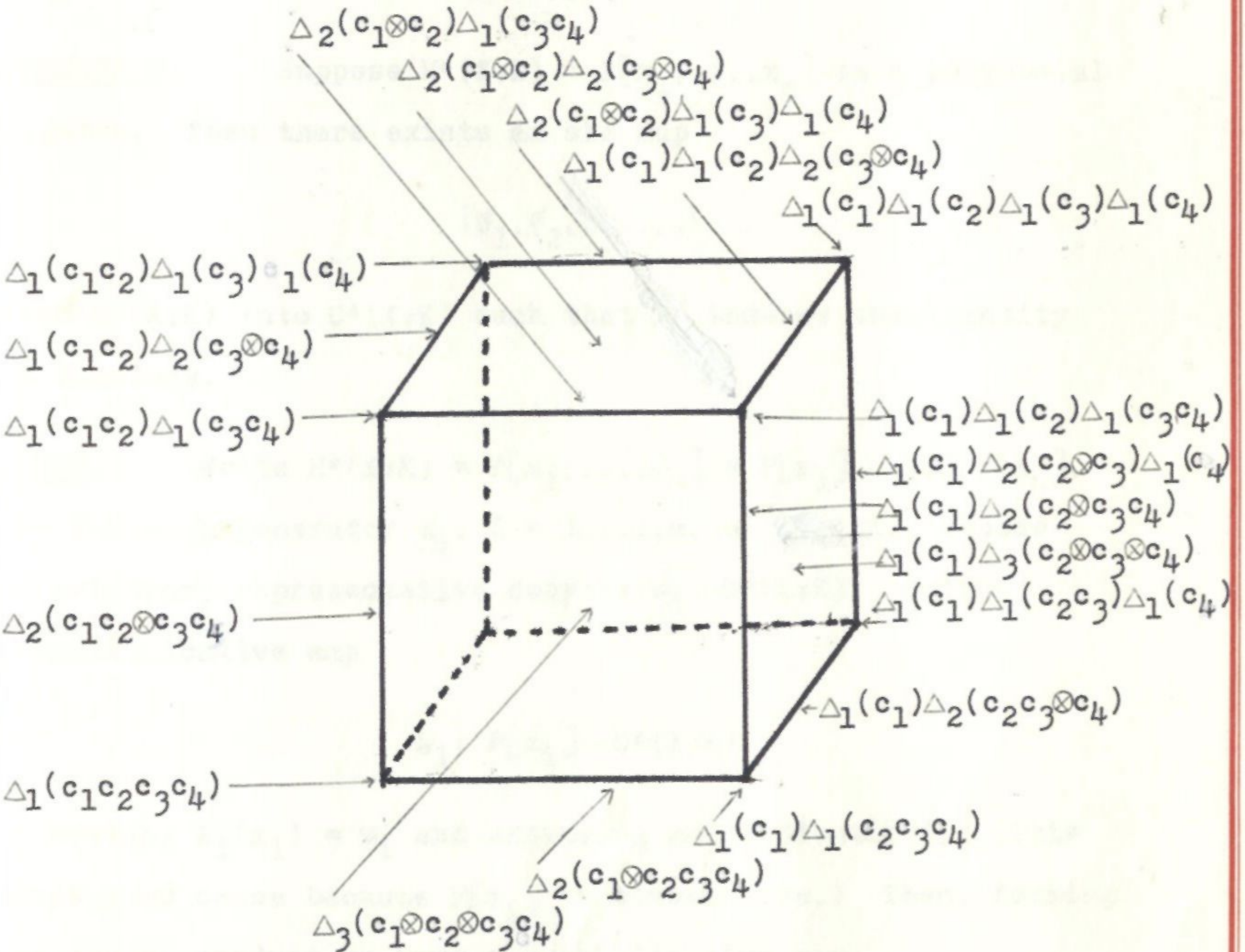
defined by

$$\nabla_4(c_1 \otimes c_2 \otimes c_3 \otimes c_4) = \Delta_1(c_1)\Delta_3(c_2 \otimes c_3 \otimes c_4) - \Delta_3(c_1 c_2 \otimes c_3 \otimes c_4)$$

$$- \Delta_2(c_1 \otimes c_2)\Delta_2(c_3 \otimes c_4) + \Delta_3(c_1 \otimes c_2 c_3 \otimes c_4)$$

$$+ \Delta_3(c_1 \otimes c_2 \otimes c_3)\Delta_1(c_4) - \Delta_3(c_1 \otimes c_2 \otimes c_3 c_4).$$

Considering the figure



We thus observe that ∇_4 is a cycle; hence ∇_4 is a boundary: That is, there exists

$$\Delta_4 : C^*(X;K) \otimes \ldots (8) \ldots \otimes C^*(X;K) \to C^*(X;K)$$

such that

(i). Δ_4 has degree -3

(ii). $(d\Delta_4 + \Delta_4 d)(c_1 \otimes c_2 \otimes c_3 \otimes c_4) = \nabla_4(c_1 \otimes c_2 \otimes c_3 \otimes c_4)$,

as desired.

Continuing in this manner, we construct all the higher Δ_n's.

<u>THEOREM 2</u>: Suppose $H^*(X;K) = P[x_1,\ldots,x_n]$ is a polynomial algebra. Then there exists an shm map

$$\{\phi_1, \phi_2, \phi_3, \ldots\}$$

from $H^*(X;K)$ into $C^*(X;K)$ such that ϕ_1 induces the identity in homology.

<u>PROOF</u>: Write $H^*(X;K) = P[x_1,\ldots,x_n] \approx P[x_1] \otimes \ldots \otimes P[x_n]$. Now for each generator x_i, $i = 1,\ldots,n$, of $H^*(X;K)$, choose an arbitrary representative cocycle $u_i \in C^*(X;K)$. Define a multiplicative map

$$\lambda_1 : P[x_i] \to C^*(X;K)$$

by setting $\lambda_i(x_i) = u_i$ and extending multiplicatively. (This makes good sense because $P[x_i]$ is commutative.) Then, forming the tensor product we have a multiplicative map

$$\lambda_1 \otimes \ldots \otimes \lambda_n : H^*(X;K) \to C^*(X;K) \otimes \ldots (n) \ldots \otimes C^*(X;K)$$

Next, let $\{\Delta_1^n, \Delta_2^n, \Delta_3^n, \ldots\}$ denote the (up to sign) shm map from $C^*(X;K)\otimes \ldots (n) \ldots \otimes C^*(X;K) \to C^*(X;K)$ defined inductively by

(1). $\{\Delta_1^1, \Delta_2^1, \Delta_3^1, \ldots\} = \{1,0,0,0,\ldots\}$;

(2). $\{\Delta_1^2, \Delta_2^2, \Delta_3^2, \ldots\} = \{\Delta_1, \Delta_2, \Delta_3, \ldots\}$;

and, having defined $\{\Delta_1^{n-1}, \Delta_2^{n-1}, \Delta_3^{n-1}, \ldots\}$, define

(n). $\{\Delta_1^n, \Delta_2^n, \Delta_3^n, \ldots\} =$

$= \{\Delta_1, \Delta_2, \Delta_3, \ldots\} \circ (\{\Delta_1^{n-1}, \Delta_2^{n-1}, \Delta_2^{n-1}, \ldots\} \otimes \{1,0,0,0,\ldots\})$.

At this point there are still sign problems. However the composition

$$\{\phi_1, \phi_2, \phi_3, \ldots\} = \{\lambda_1 \otimes \ldots \otimes \lambda_n, 0,0,0,\ldots\} \circ \{\Delta_1^n, \Delta_2^n, \Delta_3^n, \ldots\}$$

is indeed an shm map from $H^*(X;K) = P[x_1,\ldots,x_n]$ into $C^*(X;K)$: The sign discrepancy arising from $\Delta_n d$ disappears, since $H^*(X;K)$ has 0 differential. The sign discrepency arising from apparant difference between $(-1)^i$ and $\sigma(i)$ is non-existant as well, since all elements in a polynomial algebra $P[x_1,\ldots,x_n]$ have even degree unless the characteristic of K is 2 — in which case all signs are irrelevant.

Furthermore $\phi_1(x_i) = u_i$ for each $i = 1,\ldots,n$; Hence ϕ_1 induces the indentity in homology.

<u>REMARK</u>: We observe that another example of an shc algebra is $C_*(\Omega X;K)$, the chains on the loops of an H-space X. For details see Clark [8].

REMARK: We now come to a very important theorem. We shall describe an inductive algorithm for computing the terms of an shm map $\{\theta_1, \theta_2, \theta_3, \ldots\}$ from $H^*(BG;K)$ into $C^*(BG;K)$ with nice properties. The proof will be, of course, an inductive argument as well, and we shall present the first several steps of this induction in addition to the general inductive step. We apologize in advance for being so wordy; our excuse is that the arguments involved will become clearer and more natural.

THEOREM 3: Suppose $H^*(X;K) = P[x_1, \ldots, x_n]$ is a polynomial algebra. Then there exists an shm map

$$\{\theta_1, \theta_2, \theta_3, \ldots\}$$

from $H^*(X;K)$ into $C^*(X;K)$, written in terms of $\cup$- and $\cup_1$-products, such that θ_1 induces the identity in homology.

PROOF: We note first that in writing our inductive algorithm for $\{\theta_1, \theta_2, \theta_3, \ldots\}$ it will always be the case that the right hand side of any $\cup_1$-product which appears will be a single representative cocycle of $P[x_1, \ldots, x_n]$. In particular it will have even degree unless the characteristic of K is 2 — in which case all signs are again irrelevant. Thus the formulas for $\cup_1$-products are somewhat simplified. We have

(ii)'. $d(a \cup_1 b) = ab - ba - da \cup_1 b$.

(iii)'. $(ab \cup_1 c) = (-1)^{\mathrm{Deg}(a)} a(b \cup_1 c) + (a \cup_1 c)b$.

INDUCTIVE ALGORITHM: To define $\theta_1 : P[x_1, \ldots, x_n] \to C^*(X;K)$

we proceed in the usual way: For each generator x_i, $i = 1,\ldots,n$, of $H^*(X;K)$, choose an arbitrary representative cocycle $u_i \in C^*(X;K)$. Now for each monomial $x_1^{\alpha_1}\ldots x_n^{\alpha_n}$ define

$$\theta_1(x_1^{\alpha_1}\ldots x_n^{\alpha_n}) = u_1^{\alpha_1}\ldots u_n^{\alpha_n}$$

Finally, extend linearly to all of $H^*(X;K)$. (One observes that θ_1 induces the identity in homology, since $\theta_1(x_i) = u_i$, $i = 1,\ldots,n$.)

Next, assume we have written $\theta_1,\ldots,\theta_{m-1}$ in terms of $\cup$- and $\cup_1$-products. To define

$$\theta_m\colon H^*(X;K)\otimes \ldots(m)\ldots \otimes H^*(X;K) \to C^*(X;K)$$

we define first

$$\theta_m(A_1\otimes \ldots \otimes A_{m-1}\otimes x)$$

where $A_1,\ldots,A_{m-1}$ are monomials in $H^*(X;K)$ and $x = x_i$ is a generator of $H^*(X;K)$: For each $j = 1,\ldots,m-1$, write

$$A_j = B_jC_j,$$

where $B_j = x_1^{\alpha_1}\ldots x_i^{\alpha_i}$ consists of all elements of A_j with indices $\leq i$, and $C_j = x_{i+1}^{\alpha_{i+1}}\ldots x_n^{\alpha_n}$ consists of all elements of A_j with indices $> i$. Now set

$$\theta_m(A_1\otimes \ldots \otimes A_{m-1}\otimes x) = \theta_{m-1}(A_1\otimes \ldots \otimes A_{m-2}\otimes B_{m-1})[\theta_1(C_{m-1})\cup_1 u]$$

$$- \theta_{m-2}(A_1\otimes \ldots \otimes A_{m-3}\otimes B_{m-2}B_{m-1})[\theta_2(C_{m-2}\otimes C_{m-1})\cup_1 u]$$

$$+ \theta_{m-3}(A_1\otimes \ldots \otimes A_{m-4}\otimes B_{m-3}B_{m-2}B_{m-1})[\theta_3(C_{m-3}\otimes C_{m-2}\otimes C_{m-1})\cup_1 u]$$

$- \dots \pm \theta_1(B_1 \dots B_{m-1})[\theta_{m-1}(C_1 \otimes \dots \otimes C_{m-1}) \cup_1 u] =$

$\sum_{i=1}^{m-1} (-1)^{m-i-1} \theta_i(A_1 \otimes \dots \otimes A_{i-1} \otimes B_i \dots B_{m-1})[\theta_{m-i}(C_i \otimes \dots \otimes C_{m-1}) \cup_1 u].$

Now to define $\theta_m(A_1 \otimes \dots \otimes A_{m-1} \otimes A_m)$, where $A_m = x_1^{\beta_1} \dots x_n^{\beta_n}$ is a monomial in $H^*(X;K)$, we use the above rule and set

$$\theta_m(A_1 \otimes \dots \otimes A_{m-1} \otimes A_m) = \theta_m(A_1 \otimes \dots \otimes A_{m-1} \otimes x_1) u_1^{\beta_1 - 1} \dots u_n^{\beta_n} +$$

$\theta_m(A_1 \otimes \dots \otimes A_{m-2} \otimes A_{m-1} x_1 \otimes x_1) u_1^{\beta_1 - 2} \dots u_n^{\beta_n} +$

$\theta_m(A_1 \otimes \dots \otimes A_{m-2} \otimes A_{m-1} x_1^2 \otimes x_1) u_1^{\beta_1 - 3} \dots u_n^{\beta_n} + \dots +$

$\theta_m(A_1 \otimes \dots \otimes A_{m-2} \otimes A_{m-1} x_1^{\beta_1} \dots x_n^{\beta_n - 1} \otimes x_n) =$

$\sum_{i=1}^{n} \sum_{j=1}^{\beta_i} \theta_m(A_1 \otimes \dots \otimes A_{m-2} \otimes A_{m-1} x_1^{\beta_1} \dots x_i^{j-1} \otimes x_i) u_i^{\beta_i - j} \dots u_n^{\beta}$.

Finally, extend linearly to all of $H^*(X;K) \otimes \dots (m) \dots \otimes H^*(X;K)$.

<u>NOTATION</u>: To justify our definition of $\theta_m(A_1 \otimes \dots \otimes A_{m-1} \otimes x)$ we must show, inductively, that $d\theta_m(A_1 \otimes \dots \otimes A_{m-1} \otimes x)$ agrees with the appropriate version of the right-hand side of the definition of shm.

To help our exposition we shall write $d\theta_m(A_1 \otimes \dots \otimes A_{m-1} \otimes x)$ as the sum of eight components

$$I + II + III + IV + V + VI + VII + VIII,$$

where

(1). I consists of all elements of the form

$\theta_j(A_1 \otimes \dots \otimes A_j)\theta_{i-j}(A_{j+1} \otimes \dots \otimes A_{i-1} \otimes B_i \dots B_{m-1}) \cdot$

$\cdot[\theta_{m-i}(C_i \otimes \dots \otimes C_{m-1}) \cup_1 u].$

(2). II consists of all elements of the form

$$\theta_{i-1}(A_1 \otimes \ldots \otimes A_j A_{j+1} \otimes \ldots \otimes A_{i-1} \otimes B_i \ldots B_{m-1}) \cdot$$
$$\cdot [\theta_{m-i}(C_i \otimes \ldots \otimes C_{m-1}) \cup {}_1 u].$$

(3). III consists of all elements of the form

$$\theta_{i-1}(A_1 \otimes \ldots \otimes A_{i-2} \otimes A_{i-1} B_i \ldots B_{m-1}) \cdot$$
$$\cdot [\theta_{m-i}(C_i \otimes \ldots \otimes C_{m-1}) \cup {}_1 u].$$

(4). IV consists of all elements of the form

$$\theta_i(A_1 \otimes \ldots \otimes A_{i-1} \otimes B_i \ldots B_{m-1}) \theta_{m-i}(C_i \otimes \ldots \otimes C_{m-1}) u.$$

(5). V consists of all elements of the form

$$\theta_i(A_1 \otimes \ldots \otimes A_{i-1} \otimes B_i \ldots B_{m-1}) u \theta_{m-i}(C_i \otimes \ldots \otimes C_{m-1}).$$

(6). VI consists of all elements of the form

$$\theta_i(A_1 \otimes \ldots \otimes A_{i-1} \otimes B_i \ldots B_{m-1}) \theta_j(C_i \otimes \ldots \otimes C_{i+j-1}) \cdot$$
$$\cdot [\theta_{m-i-j}(C_{i+j} \otimes \ldots \otimes C_{m-1}) \cup {}_1 u].$$

(7). VII consists of all elements of the form

$$\theta_i(A_1 \otimes \ldots \otimes A_{i-1} \otimes B_i \ldots B_{m-1} [\theta_j(C_i \otimes \ldots \otimes C_{i+j-1}) \cup {}_1 u].$$
$$\cdot \theta_{m-i-j}(C_{i+j} \otimes \ldots \otimes C_{m-1}).$$

(8). VIII consist of all elements of the form

$$\theta_i(A_1 \otimes \ldots \otimes A_{i-1} \otimes B_i \ldots B_{m-1}) \cdot$$
$$\cdot [\theta_{m-i-1}(C_i \otimes \ldots \otimes C_j C_{j+1} \otimes \ldots \otimes C_{m-1}) \cup {}_1 u].$$

We shall write the other side, in turn, as the sum of four components

$$i + ii + iii + iv,$$

where

(1). i consists of all elements of the form

$\theta_{m-1}(A_1 \otimes \ldots \otimes A_k A_{k+1} \otimes \ldots \otimes A_{m-1} \otimes x)$.

(2). ii consists of all elements of the form

$\theta_k(A_1 \otimes \ldots \otimes A_k)\theta_{m-k}(A_{k+1} \otimes \ldots \otimes A_{m-1} \otimes x)$.

(3). iii consists of all elements of the form

$\theta_{m-1}(A_1 \otimes \ldots \otimes A_{m-2} \otimes A_{m-1}x)$.

(4). iv consists of all elements of the form

$\theta_{m-1}(A_1 \otimes \ldots \otimes A_{m-1})u$.

With this in mind, the general plan of attack is to prove...

$$I = ii.$$

$$II + VIII = i.$$

$$III + VI = 0.$$

$$IV = iv.$$

$$V + VII = iii.$$

In addition, we remark that we shall find it useful to prove, for each m, the general fact (called "Fact #m") that

$$\theta_m(A_1\otimes \ldots \otimes A_m) = \sum_{i=1}^{m} \theta_i(A_1\otimes \ldots \otimes A_{i-1}\otimes B_i \ldots B_m)\theta_{m-i+1}(C_i\otimes \ldots \otimes C_m).$$

So with all the above in mind, we proceed as follows:

CASE OF θ_2 GIVEN θ_1: Observe first that Fact #1 is trivial:

$$\theta_1(A_1) = \theta_1(B_1)\theta_1(C_1).$$

Now we justify the definition of $\theta_2(A_1\otimes x)$...

We have

I = 0.

II = 0.

III = 0.

IV = $+\theta_1(B_1)\theta_1(C_1)u$.

V = $-\theta_1(B_1)u\theta_1(C_1)$.

VI = 0.

VII = 0.

VIII = 0.

We also have

i = 0.

ii = 0.

iii = $-\theta_1(A_1x)$.

iv = $+\theta_1(A_1)u$.

Now we check

I = ii: Nothing to prove.

II + VIII = i: Nothing to prove.

III + VI = 0: Nothing to prove.

IV = iv: This is fact #1: $\theta_1(B_1)\theta_1(C_1)u = \theta_1(A_1)u$.

V + VII = iii: This is fact #1 applied to $A_1x = (B_1x)C_1$:

$$\theta_1(B_1)u\theta_1(C_1) = \theta_1(B_1x)\theta_1(C_1) = \theta_1(A_1x).$$

Next we justify the definiton of $\theta_2(A_1 \otimes A_2)$...

First decompose A_2 into any product

$$A_2 = A_2^{*}A_2^{**},$$

such that the indices of all elements of A_2^{*} are $\leqslant$ the indices of all elements of A_2^{**}.

Now notice that $\theta_2(A_1 \otimes A_2)$ must satisfy

$$d\theta_2(A_1 \otimes A_2) = \theta_1(A_1)\theta_1(A_2) - \theta_1(A_1A_2).$$

On the other hand $\theta_2(A_1 \otimes A_2^{*})\theta_1(A_2^{**}) + \theta_2(A_1A_2^{*} \otimes A_2^{**})$ must satisfy

$$d[\theta_2(A_1 \otimes A_2^*)\theta_1(A_2^{**}) + \theta_2(A_1A_2^* \otimes A_2^{**})] = [\theta_1(A_1)\theta_1(A_2^*)\theta_1(A_2^{**})$$

$$- \theta_1(A_1A_2^*)\theta_1(A_2^{**})] + [\theta_1(A_1A_2^*)\theta_1(A_2^{**}) - \theta_1(A_1A_2^*A_2^{**})] =$$

$$\theta_1(A_1)\theta_1(A_2) - \theta_1(A_1A_2).$$

Notice that the right hand sides of the above equations are equal. There is a certain amount of subtlety involved in what follows: By virtue of the above equality we are justified in defining

$$\theta_2(A_1 \otimes A_2) = \theta_2(A_1 \otimes x_1^{\beta_1} \ldots x_n^{\beta_n}) = \theta_2(A_1 \otimes x_1)u_1^{\beta_1 - 1} \ldots u_n^{\beta_n} + \theta_2(A_1x_1 \otimes x_1^{\beta_1 - 1} \ldots x_n^{\beta_n}),$$

except that we have not yet defined the last term. However, by exact repitition of the above argument we are justified in defining

$$\theta_2(A_1x_1 \otimes x_1^{\beta_1 - 1} \ldots x_n^{\beta_n}) = \theta_2(A_1x_1 \otimes x_1)u_1^{\beta_1 - 2} \ldots u_n^{\beta_n} + \theta_2(A_1x_1^2 \otimes x_1^{\beta_1 - 2} \ldots x_n^{\beta_n}),$$

except that we have not yet defined the last term. Continuing in this manner, we are ultimately left with the problem of defining $\theta_2(A_1x_1^{\beta_1} \ldots x_n^{\beta_n - 1} \otimes x_n)$. But this is, of course, no problem at all. This justifies the definition:

$$\theta_2(A_1 \otimes A_2) = \sum_{i=1}^{n} \sum_{j=1}^{\beta_i} \theta_2(A_1x_1^{\beta_1} \ldots x_i^{j-1} \otimes x_i)u_i^{\beta_i - j} \ldots u_n^{\beta_n}.$$

We also observe that now by definition we have

$$\theta_2(A_1 \otimes A_2) = \theta_2(A_1 \otimes A_2^*)\theta_1(A_2^{**}) + \theta_2(A_1A_2^* \otimes A_2^{**}),$$

since both sides are equal to the double summation above. In particular, this implies that

$$\theta_2(A_1 \otimes A_2) = \theta_2(A_1 \otimes B_2)\theta_1(C_2) + \theta_2(A_1B_2 \otimes C_2),$$

which is a first step towards proving Fact #2.

CASE OF θ_3 GIVEN θ_1, θ_2: In order to complete the proof of Fact #2, we first note that

$$\theta_2(A_1 \otimes C_2) = \theta_1(B_1)\theta_2(C_1 \otimes C_2)$$

is obvious. Second, applying this to A_1B_2 (= $(B_1B_2)C_1$) and C_2, we obtain

$$\theta_2(A_1B_2 \otimes C_2) = \theta_1(B_1B_2)\theta_2(C_1 \otimes C_2).$$

Adding this to the last result of the previous case (Case θ_2) we obtain

$$\theta_2(A_1 \otimes A_2) = \theta_2(A_1 \otimes B_2)\theta_1(C_2) + \theta_1(B_1B_2)\theta_2(C_1 \otimes C_2),$$

which is Fact #2.

Now we justify the definition of $\theta_3(A_1 \otimes A_2 \otimes x)$...

We have

$I = + \theta_1(A_1)\theta_1(B_2)[\theta_1(C_2) \cup_1 u].$

$II = 0.$

$III = - \theta_1(A_1B_2)[\theta_1(C_2) \cup_1 u].$

$IV = - \theta_2(A_1 \otimes B_2)\theta_1(C_2)u - \theta_1(B_1B_2)\theta_2(C_1 \otimes C_2)u.$

$V = + \theta_2(A_1 \otimes B_2)u\theta_1(C_2) + \theta_1(B_1B_2)u\theta_2(C_1 \otimes C_2).$

$VI = + \theta_1(B_1B_2)\theta_1(C_1)[\theta_1(C_2) \cup {}_1u].$

$VII = + \theta_1(B_1B_2)[\theta_1(C_1) \cup {}_1u]\theta_1(C_2).$

$VIII = - \theta_1(B_1B_2)[\theta_1(C_1C_2) \cup {}_1u].$

We also have

$i = - \theta_2(A_1A_2 \otimes x).$

$ii = + \theta_1(A_1)\theta_2(A_2 \otimes x).$

$iii = + \theta_2(A_1 \otimes A_2x).$

$iv = - \theta_2(A_1 \otimes A_2)u.$

Now we check

$I = ii$: This is by definition: $\theta_1(A_1)\theta_1(B_1)[\theta_1(C_2) \cup {}_1u] =$

$\theta_1(A_1)\theta_2(A_2 \otimes x).$

$II + VIII = i$: This is by definition:

$\theta_1(B_1B_2)[\theta_1(C_1C_2) \cup {}_1u] = \theta_2(A_1A_2 \otimes x).$

$III + VI = 0$: This is fact #1 applied to A_1B_2 $(= (B_1B_2)C_1)$:

$\theta_1(B_1B_2)\theta_1(C_1)[\theta_1(C_2) \cup {}_1u] - \theta_1(A_1B_2)[\theta_1(C_2) \cup {}_1u] =$

$[\theta_1(B_1B_2)\theta_1(C_1) - \theta_1(A_1B_2)][\theta_1(C_2) \cup {}_1u] = 0.$

$IV = iv$: This is fact #2: $\theta_2(A_1 \otimes B_2)\theta_1(C_2)u$ -

$+ \theta_1(B_1B_2)\theta_2(C_1 \otimes C_2)u = \theta_2(A_1 \otimes A_2)u.$

V + VII = iii: This is fact #2 applied to A_1 and A_2x $(= (B_2x)C_2)$:

$$\theta_2(A_1 \otimes B_2)u\theta_1(C_2) + \theta_1(B_1B_2)u\theta_2(C_1 \otimes C_2) + \theta_1(B_1B_2)[\theta_1(C_1) \cup_1 u]\theta_1(C_2)$$

$$= \theta_2(A_1 \otimes B_2)u\theta_1(C_2) + \theta_2(B_1B_2C_1 \otimes x)\theta_1(C_2) +$$

$$\theta_1(B_1(B_2x))\theta_2(C_1 \otimes C_2) = \theta_2(A_1 \otimes B_2)u\theta_1(C_2) + \theta_2(A_1B_2 \otimes x)\theta_1(C_2) +$$

$$\theta_1(B_1(B_2x))\theta_2(C_1 \otimes C_2) = \theta_2(A_1 \otimes B_2x)\theta_1(C_2) +$$

$$\theta_1(B_1(B_2x))\theta_2(C_1 \otimes C_2) = \theta_2(A_1 \otimes A_2x).$$

Next we justify the definition of $\theta_3(A_1 \otimes A_2 \otimes A_3)$

First decompose A_3 into any product

$$A_3 = A_3^* A_3^{**},$$

such that the indices of all elements of A_3^* are $\leq$ the indices of all elements of A_3^{**}.

Now notice that $\theta_3(A_1 \otimes A_2 \otimes A_3)$ must satisfy

$$d\theta_3(A_1 \otimes A_2 \otimes A_3) = \theta_1(A_1)\theta_2(A_2 \otimes A_3) - \theta_2(A_1A_2 \otimes A_3)$$

$$- \theta_2(A_1 \otimes A_2)\theta_1(A_3) + \theta_2(A_1 \otimes A_2A_3).$$

On the other hand $\theta_3(A_1 \otimes A_2 \otimes A_3^*)\theta_1(A_3^{**}) + \theta_3(A_1 \otimes A_2A_3^* \otimes A_3^{**})$ must satisfy

$$d[\theta_3(A_1 \otimes A_2 \otimes A_3^*)\theta_1(A_3^*) + \theta_3(A_1 \otimes A_2A_3^* \otimes A_3^{**})] =$$

$$[\theta_1(A_1)\theta_2(A_2 \otimes A_3^*)\theta_1(A_3^{**}) - \theta_2(A_1A_2 \otimes A_3^*)\theta_1(A_3^{**})$$

$$- \theta_2(A_1 \otimes A_2)\theta_1(A_3^*)\theta_1(A_3^{**}) + \theta_2(A_1 \otimes A_2A_3^*)\theta_1(A_3^{**})] +$$

$$[\theta_1(A_1)\theta_2(A_2A_3^* \otimes A_3^{**}) - \theta_2(A_1A_2A_3^* \otimes A_3^{**}) - \theta_2(A_1 \otimes A_2A_3^*)\theta_1(A_3^{**}) +$$

$$\theta_2(A_1 \otimes A_2A_3^*A_3^{**})] = [\theta_1(A_1)\theta_2(A_2 \otimes A_3^*)\theta_1(A_3^{**}) + \theta_1(A_1)\theta_2(A_2A_3^* \otimes A_3^{**})]$$

$$- [\theta_2(A_1A_2 \otimes A_3^*)\theta_1(A_3^{**}) + \theta_2(A_1A_2A_3^* \otimes A_3^{**})] - \theta_2(A_1 \otimes A_2)\theta_1(A_3^*)\theta_1(A_3^{**})$$

$$+ \theta_2(A_1 \otimes A_2A_3^*A_3^{**}) = \theta_1(A_1)\theta_2(A_2 \otimes A_3^*A_3^{**}) - \theta_2(A_1A_2 \otimes A_3^*A_3^{**})$$

$$- \theta_2(A_1 \otimes A_2)\theta_1(A_3^*A_3^{**}) + \theta_2(A_1 \otimes A_2A_3) = \theta_1(A_1)\theta_2(A_2 \otimes A_3)$$

$$- \theta_2(A_1A_2 \otimes A_3) - \theta_2(A_1 \otimes A_2)\theta_1(A_3) + \theta_2(A_1 \otimes A_2A_3).$$

Notice, again, that the right hand sides of the above equations are equal. So we play the same game as in the previous case (Case θ_2): By virtue of the above equality we are justified in defining

$$\theta_3(A_1 \otimes A_2 \otimes A_3) = \theta_3(A_1 \otimes A_2 \otimes x_1^{\beta_1} \ldots x_n^{\beta_n}) = \theta_3(A_1 \otimes A_2 \otimes x_1)u_1^{\beta_1-1} \ldots u_n^{\beta_n} +$$

$$+ \theta_3(A_1 \otimes A_2x_1 \otimes x_1^{\beta_1-1} \ldots x_n^{\beta_n}),$$

except that we have not yet defined the last term. However, by exact repitition of the above argument we are justified in defining

$$\theta_3(A_1 \otimes A_2x_1 \otimes x_1^{\beta_1-1} \ldots x_n^{\beta_n}) = \theta_3(A_1 \otimes A_2x_1 \otimes x_1)u_1^{\beta_1-2} \ldots u_n^{\beta_n}$$

$$+ \theta_3(A_1 \otimes A_2x_1^2 \otimes x_1^{\beta_1-2} \ldots x_n^{\beta_n}),$$

except that we have not yet defined the last term. Continuing in this manner, we are ultimately left with the problem of defining $\theta_3(A_1 \otimes A_2x_1^{\beta} \ldots x_n^{\beta_n-1} \otimes x_n)$. But this is, of course, no problem at all. This justifies the definition

$$\theta_3(A_1\otimes A_2\otimes A_3) = \sum_{i=1}^{n}\sum_{j=1}^{\beta_i} \theta_3(A_1\otimes A_2x_1^{\beta} \ldots x_i^{j-1}\otimes x_i)u_i^{\beta} \ldots u_n^{\beta} .$$

We also observe that now by definition we have

$$\theta_3(A_1\otimes A_2\otimes A_3) = \theta_3(A_1\otimes A_2\otimes A_3^*)\theta_1(A_3^{**}) + \theta_3(A_1\otimes A_2A_3^*\otimes A_3^{**}),$$

since both sides are equal to the double summation above. In particular, this implies that

$$\theta_3(A_1\otimes A_2\otimes A_3) = \theta_3(A_1\otimes A_2\otimes B_3)\theta_1(C_3) + \theta_3(A_1\otimes A_2B_3\otimes C_3),$$

which is a first step towards proving Fact #3.

<u>CASE OF θ_4 GIVEN $\theta_1, \theta_2, \theta_3$:</u> In order to complete the proof of Fact #3, we first note that

$$\theta_3(A_1\otimes A_2\otimes C_3) = \theta_2(A_1\otimes B_2)\theta_2(C_2\otimes C_3) + \theta_1(B_1B_2)\theta_3(C_1\otimes C_2\otimes C_3).$$

To see this, we first consider a special case...

$$\theta_3(A_1\otimes A_2\otimes y) = \theta_2(A_1\otimes B_2)\theta_2(C_2\otimes y) + \theta_1(B_1B_2)\theta_3(C_1\otimes C_2\otimes y),$$

where, in our notation, $y = x_j$ is a generator of $H^*(X;K)$ of index greater than the index of x (e.g., $y \in C_3$). Now write

$$C_1 = D_1E_1,$$

$$C_2 = D_2E_2,$$

where, for each $k = 1,2$, D_k consists of all elements of C_k with indices $\leq j$, and E_k consists of all elements of

C_k with indices > j. We have

$$\theta_3(A_1 \otimes A_2 \otimes y) = \theta_3(B_1D_1E_1 \otimes B_2D_2E_2 \otimes y) = \theta_2(A_1 \otimes B_2D_2)[\theta_1(E_2) \cup_1 v]$$

$$+ \theta_1(B_1D_1B_2D_2)[\theta_2(E_1 \otimes E_2) \cup_1 v] = \theta_2(A_1 \otimes B_2)\theta_1(D_2)[\theta_1(E_2) \cup_1 v] +$$

$$\theta_2(A_1 B_2 \otimes D_2)[\theta_1(E_2) \cup_1 v] - \theta_1(B_1B_2)\theta_1(D_1D_2)[\theta_2(E_1 \otimes E_2) \cup_1 v] =$$

$$\theta_2(A_1 \otimes B_2)\theta_2(C_2 \otimes y) + \theta_1(B_1B_2)\theta_2(C_1 \otimes D_2)\ [\theta_1(E_2) \cup_1 v]$$

$$- \theta_1(B_1B_2)\theta_1(D_1D_2)[\theta_2(E_1 \otimes E_2) \cup_1 v] = \theta_2(A_1 \otimes B_2)\theta_2(C_2 \otimes y) +$$

$$\theta_1(B_1B_2)\theta_2(C_1 \otimes C_2 \otimes y),$$

as desired. By repeated application of this we now have

$$\theta_3(A_1 \otimes A_2 \otimes C_3) = \theta_3(A_1 \otimes A_2 \otimes x_{i+1}^{\beta_{i+1}} \cdots x_n^{\beta_n}) =$$

$$\sum_{p=i+1}^{n} \sum_{q=1}^{\beta_p} \theta_3(A_1 \otimes A_2 x_{i+1}^{\beta_{i+1}} \cdots x_p^{q-1} \otimes x_p) u_p^{\beta_p - q} \cdots u_n^{\beta_n} =$$

$$\sum_{p=i+1}^{n} \sum_{q=1}^{\beta_p} [\theta_2(A_1 \otimes B_2)\theta_2(C_2 x_{i+1}^{\beta_{i+1}} \cdots x_p^{q-1} \otimes x_p) u_p^{\beta_p - q} \cdots u_n^{\beta_n} +$$

$$\theta_1(B_1B_2)\theta_3(C_1 \otimes C_2 x_{i+1}^{\beta_{i+1}} \cdots x_p^{q-1} \otimes x_p) u_p^{\beta_p - q} \cdots u_n^{\beta_n}] =$$

$$\theta_2(A_1 \otimes B_2)[\sum_{p=i+1}^{n} \sum_{q=1}^{\beta_p} \theta_2(C_2 x_{i+1}^{\beta_{i+1}} \cdots x_p^{q-1} \otimes x_p) u_p^{\beta_p - q} \cdots u_n^{\beta_n}] +$$

$$\theta_1(B_1B_2)[\sum_{p=i+1}^{n} \sum_{q=1}^{\beta_p} \theta_3(C_1 \otimes C_2 x_{i+1}^{\beta_{i+1}} \cdots x_p^{q-1} \otimes x_p) u_p^{\beta_p - q} \cdots u_n^{\beta_n}] =$$

$$\theta_2(A_1 \otimes B_2)\theta_2(C_2 \otimes C_3) + \theta_1(B_1B_2)\theta_3(C_1 \otimes C_2 \otimes C_3),$$

as desired. Applying this fact to A_1, A_2B_3 $(= (B_2B_3)C_2)$

and C_3, we obtain

$$\theta_3(A_1 \otimes A_2B_3 \otimes C_3) = \theta_2(A_1 \otimes B_2B_3)\theta_2(C_2 \otimes C_3) + \theta_1(B_1B_2B_3)\theta_3(C_1 \otimes C_2 \otimes C_3).$$

Adding this to the last result of the previous case (Case θ_3)

we obtain

$$\theta_3(A_1 \otimes A_2 \otimes A_3) = \theta_3(A_1 \otimes A_2 \otimes B_3)\theta_1(C_3) +$$

$$+ \theta_2(A_1 \otimes B_2 B_3)\theta_2(C_2 \otimes C_3) + \theta_1(B_1 B_2 B_3)\theta_3(C_1 \otimes C_2 \otimes C_3),$$

which is Fact #3.

Now we justify the definition of $\theta_4(A_1 \otimes A_2 \otimes A_3 \otimes x)\ldots$

We have

$$I = + \theta_1(A_1)\theta_2(A_2 \otimes B_3)[\theta_1(C_3) \cup_1 u] - \theta_2(A_1 \otimes A_2)\theta_1(B_3)[\theta_1(C_3) \cup_1 u]$$

$$- \theta_1(A_1)\theta_1(B_2 B_3)[\theta_2(C_2 \otimes C_3) \cup_1 u].$$

$$II = - \theta_2(A_1 A_2 \otimes B_3)[\theta_1(C_3) \cup_1 u].$$

$$III = + \theta_2(A_1 \otimes A_2 B_3)[\theta_1(C_3) \cup_1 u] + \theta_1(A_1 B_2 B_3)[\theta_2(C_2 \otimes C_3) \cup_1 u].$$

$$IV = + \theta_3(A_1 \otimes A_2 \otimes B_3)\theta_1(C_1)u + \theta_2(A_1 \otimes B_2 B_3)\theta_2(C_2 \otimes C_3)u +$$

$$\theta_1(B_1 B_2 B_3)\theta_3(C_1 \otimes C_2 \otimes C_3)u.$$

$$V = - \theta_3(A_1 \otimes A_2 \otimes B_3)u\theta_1(C_3) - \theta_2(A_1 \otimes B_2 B_3)u\theta_2(C_2 \otimes C_3)$$

$$- \theta_1(B_1 B_2 B_3)u\theta_3(C_1 \otimes C_2 \otimes C_3).$$

$$VI = - \theta_2(A_1 \otimes B_2 B_3)\theta_1(C_2)[\theta_1(C_3) \cup_1 u]$$

$$- \theta_1(B_1 B_2 B_3)\theta_1(C_1)[\theta_2(C_2 \otimes C_3) \cup_1 u]$$

$$- \theta_1(B_1 B_2 B_3)\theta_2(C_1 \otimes C_2)[\theta_1(C_3) \cup_1 u].$$

$$VII = - \theta_2(A_1 \otimes B_2 B_3)[\theta_1(C_2) \cup_1 u]\theta_1(C_3)$$

$- \theta_1(B_1B_2B_3)[\theta_1(C_1) \cup {}_1u]\theta_2(C_2 \otimes C_3)$

$+ \theta_1(B_1B_2B_3)[\theta_2(C_1 \otimes C_2) \cup {}_1u]\theta_1(C_3).$

$VIII = + \theta_2(A_1 \otimes B_2B_3)[\theta_1(C_2C_3) \cup {}_1u] + \theta_1(B_1B_2B_3)[\theta_2(C_1C_2 \otimes C_3) \cup {}_1u]$

$- \theta_1(B_1B_2B_3)[\theta_2(C_1 \otimes C_2C_3) \cup {}_1u].$

We also have

$i = - \theta_3(A_1A_2 \otimes A_3 \otimes x) + \theta_3(A_1 \otimes A_2A_3 \otimes x).$

$ii = + \theta_1(A_1)\theta_3(A_2 \otimes A_3 \otimes x) - \theta_2(A_1 \otimes A_2)\theta_2(A_3 \otimes x).$

$iii = - \theta_3(A_1 \otimes A_2 \otimes A_3x).$

$iv = + \theta_3(A_1 \otimes A_2 \otimes A_3)u.$

Now we check:

$I = ii$: This is by definition: $\theta_1(A_1)\theta_2(A_2 \otimes B_3)[\theta_1(C_3) \otimes {}_1u]$

$- \theta_2(A_1 \otimes A_2)\theta_1(B_3)[\theta_1(C_3) \cup {}_1u] - \theta_1(A_1)\theta_1(B_2B_3)[\theta_2(C_2 \otimes C_3) \cup {}_1u]$

$= [\theta_1(A_1)\theta_2(A_2 \otimes B_3)[\theta_1(C_3) \cup {}_1u] - \theta_1(A_1)\theta_1(B_2B_3)[\theta_2(C_2 \otimes C_3) \cup {}_1u]]$

$- \theta_2(A_1 \otimes A_2)\theta_2(A_3 \otimes x) = \theta_1(A_1)\theta_3(A_2 \otimes A_3 \otimes x) - \theta_2(A_1 \otimes A_2)\theta_3(A_3 \otimes x).$

$II + VIII = i$: This is by definition:

$- \theta_2(A_1A_2 \otimes B_3)[\theta_1(C_3) \cup {}_1u] + \theta_2(A_1 \otimes B_2B_3)[\theta_1(C_2C_3) \cup {}_1u]$

$+ \theta_1(B_1B_2B_3)[\theta_2(C_1C_2 \otimes C_3) \cup {}_1u] - \theta_1(B_1B_2B_3)[\theta_2(C_1 \otimes C_2C_3) \cup {}_1u] =$

$- [\theta_2(A_1A_2 \otimes B_3)[\theta_1(C_3) \cup {}_1u] - \theta_1(B_1B_2B_3)[\theta_2(C_1C_2 \otimes C_3) \cup {}_1u]] +$

$[\theta_2(A_1 \otimes B_2B_3)[\theta_1(C_2C_3) \cup {}_1u] - \theta_1(B_1B_2B_3)[\theta_2(C_1 \otimes C_2C_3) \cup {}_1u]]$

$= - \theta_3(A_1A_2 \otimes A_3 \otimes x) + \theta_3(A_1 \otimes A_2A_3 \otimes x).$

III + VI = 0: This is Fact #1 applied to $A_1B_2B_3$ (= $(B_1B_2B_3)C_1$) and Fact #2 applied to A_1 and A_2B_3 (=$(B_2B_3)C_2$):

$+ \theta_2(A_1 \otimes A_2B_3)[\theta_1(C_3) \cup_1 u] + \theta_1(A_1B_2B_3)[\theta_2(C_2 \otimes C_3) \cup {}_1u]$

$- \theta_2(A_1 \otimes B_2B_3)\theta_1(C_2)[\theta_1(C_3) \cup {}_1u]$

$- \theta_1(B_1B_2B_3)\theta_1(C_1)[\theta_2(C_2 \otimes C_3) \cup {}_1u]$

$- \theta_1(B_1B_2B_3)\theta_2(C_1 \otimes C_2)[\theta_1(C_3) \cup {}_1u] =$

$[\theta_2(A_1 \otimes A_2B_3) - \theta_2(A_1 \otimes B_2B_3)\theta_1(C_2) - \theta_1(B_1B_2B_3)\theta_2(C_1 \otimes C_2)] \cdot$

$\cdot [\theta_1(C_3) \cup {}_1u] +$

$[\theta_1(A_1B_2B_3) - \theta_1(B_1B_2B_3)\theta_1(C_1)][\theta_2(C_2 \otimes C_3) \cup {}_1u] = 0 + 0 = 0.$

IV = iv: This is Fact #3: $\theta_3(A_1 \otimes A_2 \otimes B_3)\theta_1(C_1)u +$

$\theta_2(A_1 \otimes B_2B_3)\theta_2(C_2 \otimes C_3)u + \theta_1(B_1B_2B_3)\theta_3(C_1 \otimes C_2 \otimes C_3)u =$

$\theta_3(A_1 \otimes A_2 \otimes A_3)u.$

V + VII = iii: This is Fact #3 applied to A_1, A_2, and A_3x (= $(B_3x)C_3$):

$\theta_3(A_1 \otimes A_2 \otimes B_3)u\theta_1(C_3) + \theta_2(A_1 \otimes B_2B_3)u\theta_2(C_2 \otimes C_3) +$

$\theta_1(B_1B_2B_3)u\theta_3(C_1 \otimes C_2 \otimes C_3) + \theta_2(A_1 \otimes B_2B_3)[\theta_1(C_2) \cup {}_1u]\theta_1(C_3) +$

$\theta_1(B_1B_2B_3)[\theta_1(C_1) \cup {}_1u]\theta_2(C_2 \otimes C_3) +$

$-\theta_1(B_1B_2B_3)[\theta_2(C_1\otimes C_2)\cup_1 u]\theta_1(C_3) = \theta_3(A_1\otimes A_2\otimes B_3)u\theta_1(C_3) +$

$[\theta_2(A_1\otimes B_2B_3)[\theta_1(C_2)\cup_1 u] + \theta_1(B_1B_2B_3)[\theta_2(C_1\otimes C_2)\cup_1 u]]\theta_1(C_3) +$

$\theta_1(A_1\otimes B_2B_3)u\theta_2(C_2\otimes C_3) + \theta_1(B_1B_2B_3)[\theta_1(C_1)\cup_1 u]\theta_2(C_2\otimes C_3) +$

$\theta_1(B_1B_2B_3)u\theta_3(C_1\otimes C_2\otimes C_3) = [\theta_3(A_1\otimes A_2\otimes B_3)u +$

$\theta_3(A_1\otimes A_2B_3\otimes x)]\theta_1(C_3) + [\theta_2(A_1\otimes B_2B_3)u + \theta_2(A_1B_2B_3\otimes x)]\theta_2(C_2\otimes C_3)$

$+ \theta_1(B_1B_2(B_3x))\theta_3(C_1\otimes C_2\otimes C_3) = \theta_3(A_1\otimes A_2\otimes B_3x)\theta_1(C_3) +$

$\theta_2(A_1\otimes B_2(B_3x))\theta_2(C_2\otimes C_3) + \theta_1(B_1B_2(B_3x))\theta_3(C_1\otimes C_2\otimes C_3) =$

$\theta_3(A_1\otimes A_2\otimes A_3x).$

Next we justify the definition of $\theta_4(A_1\otimes A_2\otimes A_3\otimes A_4)$.

First decompose A_4 into any product

$$A_4 = A_4^{*}A_4^{**},$$

such that the indices of all elements of A_4^{*} are $\leq$ the indices of all elements of A_4^{**}.

Now notice that $\theta_4(A_1\otimes A_2\otimes A_3\otimes A_4)$ must satisfy

$d\theta_4(A_1\otimes A_2\otimes A_3\otimes A_4) = \theta_1(A_1)\theta_3(A_2\otimes A_3\otimes A_4) - \theta_3(A_1A_2\otimes A_3\otimes A_4)$

$- \theta_2(A_1\otimes A_2)\theta_2(A_3\otimes A_4) + \theta_3(A_1\otimes A_2A_3\otimes A_4) + \theta_3(A_1\otimes A_2\otimes A_3)\theta_1(A_4)$

$- \theta_3(A_1\otimes A_2\otimes A_3A_4).$

On the other hand $\theta_4(A_1\otimes A_2\otimes A_3\otimes A_4^{*})\theta_1(A_4^{**}) +$

$+ \theta_4(A_1\otimes A_2\otimes A_3A_4^{*}\otimes A_4^{**})$ must satisfy

$$
\begin{aligned}
& d[\theta_4(A_1 \otimes A_2 \otimes A_3 \otimes A_4^{\#})\theta_1(A_4^{\#*}) + \theta_4(A_1 \otimes A_2 \otimes A_3A_4^{\#} \otimes A_4^{\#*})] = \\
& [\theta_1(A_1)\theta_3(A_2 \otimes A_3 \otimes A_4^{\#})\theta_1(A_4^{\#*}) - \theta_3(A_1A_2 \otimes A_3 \otimes A_4^{\#})\theta_1(A_4^{\#*}) \\
& - \theta_2(A_1 \otimes A_2)\theta_2(A_3 \otimes A_4^{\#})\theta_1(A_4^{\#*}) + \theta_3(A_1 \otimes A_2A_3 \otimes A_4^{\#})\theta_1(A_4^{\#*}) + \\
& \theta_3(A_1 \otimes A_2 \otimes A_3)\theta_1(A_4^{\#})\theta_1(A_4^{\#*}) - \theta_3(A_1 \otimes A_2 \otimes A_3A_4^{\#})\theta_1(A_4^{\#*})] + \\
& [\theta_1(A_1)\theta_3(A_2 \otimes A_3A_4^{\#} \otimes A_4^{\#*}) - \theta_3(A_1A_2 \otimes A_3A_4^{\#} \otimes A_4^{\#*}) \\
& - \theta_2(A_1 \otimes A_2)\theta_2(A_3A_4^{\#} \otimes A_4^{\#*}) + \theta_3(A_1 \otimes A_2A_3A_4^{\#} \otimes A_4^{\#*}) + \\
& \theta_3(A_1 \otimes A_2 \otimes A_3A_4^{\#})\theta_1(A_4^{\#*}) - \theta_3(A_1 \otimes A_2 \otimes A_3A_4^{\#}A_4^{\#*})] = \\
& [\theta_1(A_1)\theta_3(A_2 \otimes A_3 \otimes A_4^{\#})\theta_1(A_4^{\#*}) + \theta_1(A_1)\theta_3(A_2 \otimes A_3A_4^{\#} \otimes A_4^{\#*})] \\
& - [\theta_3(A_1A_2 \otimes A_3 \otimes A_4^{\#})\theta_1(A_4^{\#*}) + \theta_3(A_1A_2 \otimes A_3A_4^{\#} \otimes A_4^{\#*})] \\
& - [\theta_2(A_1 \otimes A_2)\theta_2(A_3 \otimes A_4^{\#})\theta_1(A_4^{\#*}) + \theta_2(A_1 \otimes A_2)\theta_2(A_3A_4^{\#} \otimes A_4^{\#*})] + \\
& [\theta_3(A_1 \otimes A_2A_3 \otimes A_4^{\#})\theta_1(A_4^{\#*}) + \theta_3(A_1 \otimes A_2A_3A_4^{\#} \otimes A_4^{\#*})] + \\
& \theta_3(A_1 \otimes A_2 \otimes A_3)\theta_1(A_4^{\#})\theta_1(A_4^{\#*}) - \theta_3(A_1 \otimes A_2 \otimes A_3A_4^{\#}A_4^{\#*}) = \\
& \theta_1(A_1)\theta_3(A_2 \otimes A_3 \otimes A_4^{\#}A_4^{\#*}) - \theta_3(A_1A_2 \otimes A_3 \otimes A_4^{\#}A_4^{\#*}) \\
& - \theta_2(A_1 \otimes A_2)\theta_2(A_3 \otimes A_4^{\#}A_4^{\#*}) + \theta_3(A_1 \otimes A_2A_3 \otimes A_4^{\#}A_4^{\#*}) + \\
& \theta_3(A_1 \otimes A_2 \otimes A_3)\theta_1(A_4) - \theta_3(A_1 \otimes A_2 \otimes A_3A_4) = \\
& \theta_1(A_1)\theta_3(A_2 \otimes A_3 \otimes A_4) - \theta_3(A_1A_2 \otimes A_3 \otimes A_4) - \theta_2(A_1 \otimes A_2)\theta_2(A_3 \otimes A_4) + \\
& \theta_3(A_1 \otimes A_2A_3 \otimes A_4) + \theta_3(A_1 \otimes A_2 \otimes A_3)\theta_1(A_4) - \theta_3(A_1 \otimes A_2 \otimes A_3A_4).
\end{aligned}
$$

Notice, once again, that the right hand sides of the above equations are equal. So we play the same game as in the previous cases; By virtue of the above equality we are justified in defining

$$\theta_4(A_1 \otimes A_2 \otimes A_3 \otimes A_4) = \theta_4(A_1 \otimes A_2 \otimes A_3 \otimes x_1^{\beta_1} \dots x_n^{\beta_n}) =$$

$$\theta_4(A_1 \otimes A_2 \otimes A_3 \otimes x_1) u_1^{\beta_1 - 1} \dots u_n^{\beta_n} + \theta_4(A_1 \otimes A_2 \otimes A_3 x_1 \otimes x_1^{\beta_1 - 1} \dots x_n^{\beta_n}),$$

except that we have not yet defined the last term. However, by exact repitition of the above argument we are justified in defining

$$\theta_4(A_1 \otimes A_2 \otimes A_3 x_1 \otimes x_1^{\beta_1 - 1} \dots x_n^{\beta_n}) =$$

$$\theta_4(A_1 \otimes A_2 \otimes A_3 x_1 \otimes x_1) u_1^{\beta_1 - 2} \dots u_n^{\beta_n} + \theta_4(A_1 \otimes A_2 \otimes A_3 x_1^2 \otimes x_1^{\beta_1 - 2} \dots x_n^{\beta_n}),$$

except that we have not yet defined the last term. Continuing in this manner, we are ultimately left with the problem of defining $\theta_4(A_1 \otimes A_2 \otimes A_3 x_1^{\beta_1} \dots x_n^{\beta_n - 1} \otimes x_n)$. But this is, of course, no problem at all. This justifies the definition

$$\theta_4(A_1 \otimes A_2 \otimes A_3 \otimes A_4) = \sum_{i=1}^{n} \sum_{j=1}^{\beta_i} \theta_4(A_1 \otimes A_2 \otimes A_3 x_1^{\beta_1} \dots x_i^{j-1} \otimes x_i) u_i^{\beta_i - j} \dots u_n^{\beta_n}.$$

We also observe that now by definition we have

$$\theta_4(A_1 \otimes A_2 \otimes A_3 \otimes A_4) = \theta_4(A_1 \otimes A_2 \otimes A_3 \otimes A_4^*) \theta_1(A_4^{**}) +$$

$$+ \theta_4(A_1 \otimes A_2 \otimes A_3 A_4^* \otimes A_4^{**}),$$

since both sides are equal to the double summation above. In particular, this implies that

$$\theta_4(A_1 \otimes A_2 \otimes A_3 \otimes A_4) = \theta_4(A_1 \otimes A_2 \otimes A_3 \otimes B_4) \theta_1(C_4) +$$

$$+ \theta_4(A_1 \otimes A_2 \otimes A_3 B_4 \otimes C_4),$$

which is a first step towards proving Fact #4.

CASE OF θ_m GIVEN $\theta_1,\ldots,\theta_{m-1}$: We observe that we know by induction (from the last result of Case θ_{m-1}) that

$$\theta_{m-1}(A_1\otimes\ldots\otimes A_{m-1}) = \theta_{m-1}(A_1\otimes\ldots\otimes A_{m-2}\otimes B_{m-1})\theta_1(C_{m-1})$$
$$+ \theta_{m-1}(A_1\otimes\ldots\otimes A_{m-2}B_{m-1}\otimes C_{m-1});$$

this is a first step towards proving Fact #(m-1).

In order to complete the proof of Fact #(m-1), we first note that

$$\theta_{m-1}(A_1\otimes\ldots\otimes A_{m-2}\otimes C_{m-1}) = \sum_{i=1}^{m-2}\theta_i(A_1\otimes\ldots\otimes A_{i-1}\otimes B_i\ldots B_{m-2})\cdot$$
$$\cdot\,\theta_{m-i}(C_i\otimes\ldots\otimes C_{m-1}).$$

To see this, we consider again a special case first...

$$\theta_{m-1}(A_1\otimes\ldots\otimes A_{m-2}\otimes y) = \sum_{i=1}^{m-2}\theta_i(A_1\otimes\ldots\otimes A_{i-1}\otimes B_i\ldots B_{m-2})\cdot$$
$$\cdot\,\theta_{m-i}(C_i\otimes\ldots\otimes C_{m-2}\otimes y),$$

where, in our notation, $y = x_j$ is a generator of $H^*(X;K)$ of index greater than the index of x (e.g., $y \in C_{m-1}$). Now write

$$C_k = D_kE_k$$

for each $k = 1,\ldots,m-2$, where D_k consists of all elements of C_k with indices $\leq j$, and E_k consists of all elements of C_k with indices $> j$. We have

$$\theta_{m-1}(A_1\otimes\ldots\otimes A_{m-2}\otimes y) = \theta_{m-1}(B_1D_1E_1\otimes\ldots\otimes B_{m-2}D_{m-2}E_{m-2}\otimes y) =$$

$$\sum_{i=1}^{m-2}(-1)^{m-i}\theta_i(A_1\otimes\ldots\otimes A_{i-1}\otimes B_iD_i\ldots B_{m-2}D_{m-2})[\theta_{m-i-1}(E_i\otimes\ldots\otimes E_{m-2})\cup_1 v]$$

$$= \sum_{i=1}^{m-2}\sum_{k=1}^{i}(-1)^{m-i}[\theta_k(A_1\otimes\ldots\otimes A_{k-1}\otimes B_k\ldots B_{i-1}B_i\ldots B_{m-2})\cdot$$

$$\cdot\theta_{i-k+1}(C_k\otimes\ldots\otimes C_{i-1}\otimes D_i\ldots D_{m-2})][\theta_{m-i-1}(E_i\otimes\ldots\otimes E_{m-2})\cup_1 v] =$$

$$\sum_{i=1}^{m-2}\sum_{k=1}^{i}(-1)^{m-i}\theta_k(A_1\otimes\ldots\otimes A_{k-1}\otimes B_k\ldots B_{m-2})\cdot$$

$$\cdot[\theta_{i-k+1}(C_k\otimes\ldots\otimes C_{i-1}\otimes D_i\ldots D_{m-2})[\theta_{m-i-1}(E_i\otimes\ldots\otimes E_{m-2})\cup_1 v]] =$$

$$\sum_{k=1}^{m-2}\theta_k(A_1\otimes\ldots\otimes A_{k-1}\otimes B_k\ldots B_{m-2})\theta_{m-k}(C_k\otimes\ldots\otimes C_{m-2}\otimes y),$$

as desired. By repeated application of this we now have

$$\theta_{m-1}(A_1\otimes\ldots\otimes A_{m-2}\otimes C_{m-1}) = \theta_{m-1}(A_1\otimes\ldots\otimes A_{m-2}\otimes x_{i+1}^{\beta_{i+1}}\ldots x_n^{\beta_n}) =$$

$$\sum_{p=i+1}^{n}\sum_{q=1}^{\beta_p}\theta_{m-1}(A_1\otimes\ldots\otimes A_{m-3}\otimes A_{m-2}x_{i+1}^{\beta_{i+1}}\ldots x_p^{q-1}\otimes x_p)u_p^{\beta_p-q}\ldots u_n^{\beta_n} =$$

$$\sum_{p=i+1}^{n}\sum_{q=1}^{\beta_p}\sum_{k=1}^{m-2}\theta_k(A_1\otimes\ldots\otimes A_{k-1}\otimes B_k\ldots B_{m-2})\theta_{m-k}(C_k\otimes\ldots\otimes C_{m-3}\otimes$$

$$\otimes C_{m-2}x_{i+1}^{\beta_{i+1}}\ldots x_p^{q-1}\otimes x_p)u_p^{\beta_p-q}\ldots u_n^{\beta_n} =$$

$$\sum_{k=1}^{m-2}\sum_{p=i+1}^{n}\sum_{q=1}^{\beta_p}\theta_k(A_1\otimes\ldots\otimes A_{k-1}\otimes B_k\ldots B_{m-2})\theta_{m-k}(C_k\otimes\ldots\otimes C_{m-3}\otimes$$

$$\otimes C_{m-2}x_{i+1}^{\beta_{i+1}}\ldots x_p^{q-1}\otimes x_p)u_p^{\beta_p-q}\ldots u_n^{\beta_n} =$$

$$\sum_{k=1}^{m-2}\theta_k(A_1\otimes\ldots\otimes A_{k-1}\otimes B_k\ldots B_{m-2})\theta_{m-k}(C_k\otimes\ldots\otimes C_{m-1}),$$

as desired. Applying this fact to $A_1,\ldots,A_{m-3}$, $A_{m-2}B_{m-1}$ $(=(B_{m-2}B_{m-1})C_{m-2})$ and C_{m-1}, we obtain

$$\theta_{m-1}(A_1\otimes\ldots\otimes A_{m-2}B_{m-1}\otimes C_{m-1}) = \sum_{i=1}^{m-2}\theta_i(A_1\otimes\ldots\otimes A_{i-1}\otimes B_i\ldots B_{m-1})\cdot$$

$$\cdot\ \theta_{m-i}(C_i\otimes\ldots\otimes C_{m-1}).$$

Adding this to the assumed result mentioned above, we obtain

$$\theta_{m-1}(A_1 \otimes \dots \otimes A_{m-2} \otimes C_{m-1}) = \sum_{i=1}^{m-1} \theta_i(A_1 \otimes \dots \otimes A_{i-1} \otimes B_i \dots B_{m-1}) \cdot$$
$$\cdot\, \theta_{m-i}(C_i \otimes \dots \otimes C_{m-1}),$$

which is Fact #(m-1).

Now we justify the definition of $\theta_m(A_1 \otimes \dots \otimes A_{m-1} \otimes x)$...

We have

$$I = \sum_{i=1}^{m-1} \sum_{j=1}^{i-1} (-1)^{m-i+j} \theta_j(A_1 \otimes \dots \otimes A_j)\, \theta_{i-j}(A_{j+1} \otimes \dots \otimes A_{i-1} \otimes B_i \dots B_{m-1}) \cdot$$
$$\cdot [\theta_{m-i}(C_i \otimes \dots \otimes C_{m-1}) \cup {}_1 u].$$

$$II = \sum_{i=2}^{m-1} \sum_{j=1}^{i-2} (-1)^{m-i+j+1} \theta_{i-1}(A_1 \otimes \dots \otimes A_j A_{j+1} \otimes \dots \otimes A_{i-1} \otimes B_i \dots B_{m-1}) \cdot$$
$$\cdot [\theta_{m-i}(C_i \otimes \dots \otimes C_{m-1}) \cup {}_1 u].$$

$$III = \sum_{i=2}^{m-1} (-1)^m \theta_{i-1}(A_1 \otimes \dots \otimes A_{i-2} \otimes A_{i-1} B_i \dots B_{m-1}) \cdot$$
$$\cdot [\theta_{m-i}(C_i \otimes \dots \otimes C_{m-1}) \cup {}_1 u].$$

$$IV = \sum_{i=1}^{m-1} (-1)^m \theta_i(A_1 \otimes \dots \; A_{i-1} \otimes B_i \dots B_{m-1})\, \theta_{m-i}(C_i \otimes \dots \otimes C_{m-1}) u.$$

$$V = \sum_{i=1}^{m-1} (-1)^{m+1} \theta_i(A_1 \otimes \dots \otimes A_{i-1} \otimes B_i \dots B_{m-1})\, u\, \theta_{m-i}(C_i \otimes \dots \otimes C_{m-1}).$$

$$VI = \sum_{i=1}^{m-2} \sum_{j=1}^{m-i-1} (-1)^{m+1} \theta_i(A_1 \otimes \dots \otimes A_{i-1} \otimes B_i \dots B_{m-1})\, \theta_j(C_i \otimes \dots \otimes C_{i+j-1}) \cdot$$
$$\cdot [\theta_{m-i-j}(C_{i+j} \otimes \dots \otimes C_{m-1}) \cup {}_1 u].$$

$$VII = \sum_{i=1}^{m-2} \sum_{j=1}^{m-i-1} (-1)^{m+j} \theta_i(A_1 \otimes \dots \otimes A_{i-1} \otimes B_i \dots B_{m-1}) [\theta_j(C_i \otimes \dots \otimes C_{i+j-1}) \cup {}_1 u] \cdot$$
$$\cdot\, \theta_{m-i-j}(C_{i+j} \otimes \dots \otimes C_{m-1}).$$

$$VIII = \sum_{i=1}^{m-2}\sum_{j=i}^{m-2}(-1)^{m-i+j+1}\theta_i(A_1\otimes \ldots\otimes A_{i-1}\otimes B_i\ldots B_{m-1})\cdot$$

$$\cdot[\theta_{m-i-1}(C_i\otimes \ldots\otimes C_jC_{j+1}\otimes \ldots\otimes C_{m-1})\cup {}_1u].$$

We also have

$$i = \sum_{i=1}^{m-2}(-1)^i\theta_{m-1}(A_1\otimes \ldots\otimes A_iA_{i+1}\otimes \ldots\otimes A_{m-1}\otimes x).$$

$$ii = \sum_{i=1}^{m-2}(-1)^{i+1}\theta_i(A_1\otimes \ldots\otimes A_i)\theta_{m-i}(A_{i+1}\otimes \ldots\otimes A_{m-1}\otimes x).$$

$$iii = (-1)^{m-1}\theta_{m-1}(A_1\otimes \ldots\otimes A_{m-2}\otimes A_{m-1}x).$$

$$iv = (-1)^m\theta_{m-1}(A_1\otimes \ldots\otimes A_{m-1})u.$$

Now we check:

$I = ii$: This is by definition: $\sum_{i=1}^{m-1}\sum_{j=1}^{i-1}(-1)^{m-i+j}\theta_j(A_1\otimes \ldots\otimes A_j)\cdot$

$\cdot\theta_{i-j}(A_{j+1}\otimes \ldots\otimes A_{i-1}\otimes B_i\ldots B_{m-1})[\theta_{m-i}(C_i\otimes \ldots\otimes C_{m-1})\cup {}_1u] =$

$\sum_{j=1}^{m-2}\sum_{i=j+1}^{m-1}(-1)^{j+1}(-1)^{m-i+1}\theta_j(A_1\otimes \ldots\otimes A_j)\theta_{i-j}(A_{j+1}\otimes \ldots\otimes A_{i-1}\otimes B_i\ldots B_{m-1})\cdot$

$\cdot[\theta_{m-i}(C_i\otimes \ldots\otimes C_{m-1})\cup {}_1u] = (-1)^{j+1}$

$\sum_{j=1}^{m-2}(-1)^{j+1}\theta_j(A_1\otimes \ldots\otimes A_j)\theta_{m-j}(A_{j+1}\otimes \ldots\otimes A_{m-1}\otimes x).$

$II + VIII = i$: This is by definition:

$\sum_{i=2}^{m-1}\sum_{j=1}^{i-2}(-1)^{m-i+j+1}\theta_{i-1}(A_1\otimes \ldots\otimes A_jA_{j+1}\otimes \ldots\otimes A_{i-1}\otimes B_i\ldots B_{m-1})\cdot$

$\cdot[\theta_{m-i}(C_i\otimes \ldots\otimes C_{m-1})\cup {}_1u] +$

$\sum_{i=1}^{m-2}\sum_{j=i}^{m-2}(-1)^{m-i+j+1}\theta_i(A_1\otimes \ldots\otimes A_{i-1}\otimes B_i\ldots B_{m-1})\cdot$

$\cdot[\theta_{m-i-1}(C_i\otimes \ldots\otimes C_jC_{j+1}\otimes \ldots\otimes C_{m-1})\cup {}_1u] =$

$$\sum_{j=1}^{m-2} (-1)^j \Big[\sum_{i=j+1}^{m-1} (-1)^{m-i+1} \theta_{i-1}(A_1 \otimes \ldots \otimes A_j A_{j+1} \otimes \ldots \otimes A_{i-1} \otimes B_i \ldots B_{m-1}) \cdot$$

$$\cdot [\theta_{m-i}(C_i \otimes \ldots \otimes C_{m-1}) \cup_1 u] + \sum_{i=1}^{j} (-1)^{m-i+1} \theta_i(A_1 \otimes \ldots \otimes A_{i-1} \otimes B_i \ldots B_{m-1}) \cdot$$

$$\cdot [\theta_{m-i-1}(C_i \otimes \ldots \otimes C_j C_{j+1} \otimes \ldots \otimes C_{m-1}) \cup_1 u]\Big] =$$

$$\sum_{j=1}^{m-2} (-1)^j \theta_{m-1}(A_1 \otimes \ldots \otimes A_j A_{j+1} \otimes \ldots \otimes A_{m-1} \otimes x).$$

III + VI = 0: This is Fact #k applied to $A_1, A_2, \ldots, A_{k-1}$, and $A_k B_{k+1} \ldots B_{m-1}$ $(= (B_k \ldots B_{m-1}) C_k)$ for each $k = 1, \ldots, m-2$:

$$\sum_{i=2}^{m-1} (-1)^m \theta_{i-1}(A_1 \otimes \ldots \otimes A_{i-2} \otimes A_{i-1} B_i \ldots B_{m-1}) [\theta_{m-i}(C_i \otimes \ldots \otimes C_{m-1}) \cup_1 u] +$$

$$\sum_{i=1}^{m-2} \sum_{j=1}^{m-i-1} (-1)^{m+1} \theta_i(A_1 \otimes \ldots \otimes A_{i-1} \otimes B_i \ldots B_{m-1}) \theta_j(C_i \otimes \ldots \otimes C_{i+j-1}) \cdot$$

$$\cdot [\theta_{m-i-j}(C_{i+j} \otimes \ldots \otimes C_{m-1}) \cup_1 u] =$$

$$\sum_{k=1}^{m-2} (-1)^m \theta_k(A_1 \otimes \ldots \otimes A_{k-1} \otimes A_k B_{k+1} \ldots B_{m-1}) [\theta_{m-k-1}(C_{k+1} \otimes \ldots \otimes C_{m-1}) \cup_1 u]$$

$$- \sum_{p=1}^{m-2} \sum_{k=p}^{m-2} (-1)^m \theta_p(A_1 \otimes \ldots \otimes A_{p-1} \otimes B_p \ldots B_{m-1}) \theta_{k-p+1}(C_p \otimes \ldots \otimes C_k) \cdot$$

$$\cdot [\theta_{m-k-1}(C_{k+1} \otimes \ldots \otimes C_{m-1}) \cup_1 u] =$$

$$\sum_{k=1}^{m-2} (-1)^m [\theta_k(A_1 \otimes \ldots \otimes A_{k-1} \otimes A_k B_{k+1} \ldots B_{m-1})$$

$$- \sum_{p=1}^{k} \theta_p(A_1 \otimes \ldots \otimes A_{p-1} \otimes B_p \ldots B_{m-1}) \theta_{k-p+1}(C_p \otimes \ldots \otimes C_k) \cdot$$

$$\cdot [\theta_{m-k-1}(C_{k+1} \otimes \ldots \otimes C_{m-1}) \cup_1 u]] = \sum_{k=1}^{m-2} 0 = 0.$$

IV = iv: This is Fact #(m-1):

$$\sum_{i=1}^{m-1} (-1)^m \theta_i(A_1 \otimes \ldots \otimes A_{i-1} \otimes B_i \ldots B_{m-1}) \theta_{m-i}(C_i \otimes \ldots \otimes C_{m-1}) u =$$

$$(-1)^m \theta_{m-1}(A_1 \otimes \ldots \otimes A_{m-1}) u.$$

V + VII = iii: This is Fact #(m-1) applied to $A_1, \ldots, A_{m-2}$, and $A_{m-1}x$ $(= (B_{m-1}x)C_{m-1})$:

$$\sum_{i=1}^{m-1} (-1)^{m+1}\theta_i(A_1\otimes \ldots\otimes A_{i-1}\otimes B_i\ldots B_{m-1})u\theta_{m-i}(C_i\otimes \ldots\otimes C_{m-1}) +$$

$$\sum_{i=1}^{m-2}\sum_{j=1}^{m-i-1}(-1)^{m-j}\theta_i(A_1\otimes \ldots\otimes A_{i-1}\otimes B_i\ldots B_{m-1})[\theta_j(C_i\otimes \ldots\otimes C_{i+j-1})\cup {}_1u]\cdot$$

$$\cdot\ \theta_{m-i-j}(C_{i+j}\otimes \ldots\otimes C_{m-1}) =$$

$$\sum_{k=1}^{m-1} (-1)^{m+1}\theta_k(A_1\otimes \ldots\otimes A_{k-1}\otimes B_k\ldots B_{m-1})u\theta_{m-k}(C_k\otimes \ldots\otimes C_{m-1}) +$$

$$\sum_{i=1}^{m-2}\sum_{k=i+1}^{m-1}(-1)^{m-k+i}\theta_i(A_1\otimes \ldots\otimes A_{i-1}\otimes B_i\ldots B_{m-1})\cdot$$

$$\cdot[\theta_{k-i}(C_i\otimes \ldots\otimes C_{k-1})\cup {}_1u]\theta_{m-k}(C_k\otimes \ldots\otimes C_{m-1}) =$$

$$\sum_{k=1}^{m-1} (-1)^{m-1}\theta_k(A_1\otimes \ldots\otimes A_{k-1}\otimes B_k\ldots B_{m-1})u\theta_{m-k}(C_k\otimes \ldots\otimes C_{m-1}) +$$

$$\sum_{k=1}^{m-1}\sum_{i=1}^{k-1}(-1)^{m-1}(-1)^{k-i-1}\theta_i(A_1\otimes \ldots\otimes A_{i-1}\otimes B_i\ldots B_{k-1}B_k\ldots B_{m-1})\cdot$$

$$\cdot[\theta_{k-i}(C_i\otimes \ldots\otimes C_{k-1})\cup {}_1u]\theta_{m-k}(C_k\otimes \ldots\otimes C_{m-1}) =$$

$$\sum_{k=1}^{m-1} (-1)^{m-1}\theta_k(A_1\otimes \ldots\otimes A_{k-1}\otimes B_k\ldots B_{m-1})u\theta_{m-k}(C_k\otimes \ldots\otimes C_{m-1}) +$$

$$\sum_{k=1}^{m-1} (-1)^{m-1}\theta_k(A_1\otimes \ldots\otimes A_{k-1}B_k\ldots B_{m-1}\otimes x)\theta_{m-k}(C_k\otimes \ldots\otimes C_{m-1}) =$$

$$\sum_{k=1}^{m-1} (-1)^{m-1}\theta_k(A_1\otimes \ldots\otimes A_{k-1}\otimes B_k\ldots B_{m-2}(B_{m-1}x))\theta_{m-k}(C_k\otimes \ldots\otimes C_{m-1}) =$$

$$= (-1)^{m-1}\theta_{m-1}(A_1\otimes \ldots\otimes A_{m-2}\otimes A_{m-1}x).$$

Next we justify the definition of $\theta_m(A_1\otimes \ldots\otimes A_m)$

First decompose A_m into any product

$$A_m = A_m^*A_m^{**},$$

such that the indices of all elements of A_m^* are $\leq$ the indices of all elements of A_m^{**}.

Now notice that $\theta_m(A_1\otimes \ldots \otimes A_m)$ must satisfy

$$d\theta_m(A_1\otimes \ldots \otimes A_m) = \sum_{i=1}^{m-1}(-1)^i[\theta_{m-1}(A_1\otimes \ldots \otimes A_iA_{i+1}\otimes \ldots \otimes A_m)$$
$$- \theta_i(A_1\otimes \ldots \otimes A_i)\theta_{m-i}(A_{i+1}\otimes \ldots \otimes A_m)].$$

On the other hand $\theta_m(A_1\otimes \ldots \otimes A_{m-1}\otimes A_m^*)\theta_1(A_m^{**}) +$ $\theta_m(A_1\otimes \ldots \otimes A_{m-1}A_m^*\otimes A_m^{**})$ must satisfy

$$d[\theta_m(A_1\otimes \ldots \otimes A_{m-1}\otimes A_m^*)\theta_1(A_m^{**}) + \theta_m(A_1\otimes \ldots \otimes A_{m-1}A_m^*\otimes A_m^{**})] =$$
$$\sum_{i=1}^{m-2}(-1)^i[\theta_m(A_1\otimes \ldots \otimes A_iA_{i+1}\otimes \ldots \otimes A_{m-1}\otimes A_m^*)\theta_1(A_m^{**}) +$$
$$\theta_m(A_1\otimes \ldots \otimes A_iA_{i+1}\otimes \ldots \otimes A_{m-1}A_m^*\otimes A_m^{**})$$
$$- \theta_i(A_1\otimes \ldots \otimes A_i)\theta_{m-i}(A_{i+1}\otimes \ldots \otimes A_{m-1}\otimes A_m^*)\theta_1(A_m^{**})$$
$$- \theta_i(A_1\otimes \ldots \otimes A_i)\theta_{m-i}(A_{i+1}\otimes \ldots \otimes A_{m-1}A_m^*\otimes A_m^{**})] +$$
$$(-1)^{m-1}[\theta_{m-1}(A_1\otimes \ldots \otimes A_{m-1}A_m^*)\theta_1(A_m^{**}) + \theta_{m-1}(A_1\otimes \ldots \otimes A_{m-1}A_m^*A_m^{**})$$
$$- \theta_{m-1}(A_1\otimes \ldots \otimes A_{m-1})\theta_1(A_m^*)\theta_1(A_m^{**}) - \theta_{m-1}(A_1\otimes \ldots \otimes A_{m-1}A_m^*)\theta_1(A_m^{**})] =$$
$$\sum_{i=1}^{m-2}(-1)^i[\theta_{m-1}(A_1\otimes \ldots \otimes A_iA_{i+1}\otimes \ldots \otimes A_m)$$
$$- \theta_i(A_1\otimes \ldots \otimes A_i)\theta_{m-i}(A_{i+1}\otimes \ldots \otimes A_m)] +$$
$$(-1)^{m-1}[\theta_{m-1}(A_1\otimes \ldots \otimes A_{m-2}\otimes A_{m-1}A_m) - \theta_{m-1}(A_1\otimes \ldots \otimes A_{m-1})\theta_1(A_m)]$$
$$= \sum_{i=1}^{m-1}(-1)^i[\theta_{m-1}(A_1\otimes \ldots \otimes A_iA_{i+1}\otimes \ldots \otimes A_m)$$
$$- \theta_i(A_1\otimes \ldots \otimes A_i)\theta_{m-i}(A_{i+1}\otimes \ldots \otimes A_m)] .$$

Notice, one last time, that the right hand sides of the above equations are equal. So we play the same game as in all the previous cases: By virtue of the above equality we are justified in defining

$$\theta_m(A_1\otimes \ldots \otimes A_{m-1}\otimes A_m) = \theta_m(A_1\otimes \ldots \otimes A_{m-1}\otimes x_1^{\beta_1} \ldots x_n^{\beta_n}) =$$

$$\theta_m(A_1\otimes \ldots \otimes A_{m-1}\otimes x_1)u_1^{\beta_i-1}\ldots u_n^{\beta_n} + \theta_m(A_1\otimes \ldots \otimes A_{m-1}x_1\otimes x_1^{\beta_i-1}\ldots x_n^{\beta_n}),$$

except that we have not yet defined the last term. However, by exact repitition of the above argument we are justified in defining

$$\theta_m(A_1\otimes \ldots \otimes A_{m-1}x_1\otimes x_1^{\beta_i-1}\ldots x_n^{\beta_n}) =$$

$$\theta_m(A_1\otimes \ldots \otimes A_{m-1}x_1\otimes x_1)u_1^{\beta_i-2}\ldots u_n^{\beta_n} + \theta_m(A_1\otimes \ldots \otimes A_{m-1}x_1^2\otimes x_1^{\beta_i-2}\ldots x_n^{\beta_n}),$$

except that we have not yet defined the last term. Continuing in this manner, we are ultimately left with the problem of defining $\theta_m(A_1\otimes \ldots \otimes A_{m-1}x_1^{\beta_1}\ldots x_n^{\beta_n-1}\otimes x_n)$. But this is, of course, no problem at all. The justifies the definition

$$\theta_m(A_1\otimes \ldots \otimes A_m) = \sum_{i=1}^{n} \sum_{j=1}^{\beta_i} \theta_m(A_1\otimes \ldots \otimes A_{m-1}x_1^{\beta} \ldots x_i^{j-1}\otimes x_i)u_1^{\beta}\ldots u_n^{\beta}).$$

We also observe that now by definition we have

$$\theta_m(A_1\otimes \ldots \otimes A_m) = \theta_m(A_1\otimes \ldots \otimes A_{m-1}\otimes A_m^*)\theta_1(A_m^{**}) +$$

$$+ \theta_m(A_1\otimes \ldots \otimes A_{m-1}A_m^*\otimes A_m^{**}),$$

since both sides are equal to the double summation above. In particular, this implies that

$$\theta_m(A_1\otimes \ldots \otimes A_m) = \theta_m(A_1\otimes \ldots \otimes A_{m-1}\otimes B_m)\theta_1(C_m)$$

$$+ \theta_m(A_1\otimes \ldots \otimes A_{m-1}B_m\otimes C_m),$$

which is a first step towards proving Fact #m.

This completes the proof of Theorem 3.

REMARK: We observe that each term in the quantity $\theta_m(A_1\otimes\ldots\otimes A_m)$ contains precisely m-1 $\cup_1$-products. In particular, unless m = 1, each term in $\theta_m(A_1\otimes\ldots\otimes A_m)$ contains at least one $\cup_1$-product. This will be important in what follows.

EXAMPLE: Suppose $P[x_1,\ldots,x_n]$ is a polynomial algebra over K, where $n \geqslant 5$. We compute $\theta_4(x_5\otimes x_1x_2x_3x_4\otimes x_2\otimes x_1)$, a randomly selected example:

$$\theta_4(x_5\otimes x_1x_2x_3x_4\otimes x_2\otimes x_1) = \theta_3(x_5\otimes x_1x_2x_3x_4\otimes 1)[\theta_1(x_2)\cup_1 u_1]$$

$$- \theta_2(x_5\otimes x_1)[\theta_2(x_2x_3x_4\otimes x_2)\cup_1 u_1] + \theta_1(x_1)[\theta_3(x_5\otimes x_2x_3x_4\otimes x_2)\cup_1 u_1]$$

$$= 0 - [u_5\cup_1 u_1][[\theta_1(x_2)[\theta_1(x_3x_4)\cup_1 u_2]]\cup_1 u_1] +$$

$$u_1[[\theta_2(x_5\otimes x_2)[\theta_1(x_3x_4)\cup_1 u_2] - \theta_1(x_2)[\theta_2(x_5\otimes x_3x_4)\cup_1 u_2]]\cup_1 u_1] =$$

$$- [u_5\cup_1 u_1][[u_2[u_3u_4\cup_1 u_2]]\cup_1 u_1] + u_1[[[u_5\cup_1 u_2][u_3u_4\cup_1 u_2]]\cup_1 u_1]$$

$$- u_1[[[\theta_2(x_5\otimes x_3)u_4 + \theta_2(x_3x_5\otimes x_4)]\cup_1 u_2]\cup_1 u_1] =$$

$$u_1[[[u_5\cup_1 u_2][u_3u_4\cup_1 u_2]]\cup_1 u_1] - [u_5\cup_1 u_1][[u_2[u_3u_4\cup_1 u_2]]\cup_1 u_1]$$

$$- u_1[[[u_5\cup_1 u_3]u_4\cup_1 u_2]\cup_1 u_1] - u_1[[[u_3[u_5\cup_1 u_4]]\cup_1 u_2]\cup_1 u_1].$$

Notice that each term contains preciesely 3 $\cup_1$-products, as in the remark above. We can reduce still further, if we desired, by using the Hirsch formula.

REMARK: It is known that $H^*(BG;K)$ is a polynomial algebra in the cases

(i). K has characteristic 0;

(ii). K has characteristic p and $H_*(G;K)$ has no p-torsion.

Thus in these cases Theorems 2 and 3 apply to give shm maps

$$\{\phi_1, \phi_2, \phi_3, \ldots\}$$

and particularly

$$\{\theta_1, \theta_2, \theta_3, \ldots\}$$

from $H^*(BG;K)$ into $C^*(BG;K)$ with all the desired properties.

II. 3. THE COHOMOLOGY OF HOMOGENEOUS SPACES:

We have finally developed nearly all the machinary needed to state and prove the major theorem in this chapter, which is the following result on the cohomology of homogeneous spaces:

THEOREM 1: Let G be a compact, connected Lie group and H a compact, connected subgroup of G. Form the homogeneous space G/H. Suppose that either

(i). K has characteristic 0,

or

(ii). K has characteristic p, and $H_*(G;K)$, $H_*(H;K)$ have no p-torsion.

Then, regarding K as a right $H^*(BG;K)$-module via augmentation, and $H^*(BH;K)$ as a left $H^*(BG;K)$-module via the natural map $f^*:H^*(BG;K)\rightarrow H^*(BH;K)$, we have a module isomorphism

$$H^*(G/H;K) \approx \mathrm{tor}_{H^*(BG;K)}(K,H^*(BH;K)).$$

REMARK: The proof of Theorem 1 is based on several important results, and we delay the proof until these have been introduced.

The first such result is due to Eilenberg and Moore [11][12]. Suppose we are given a Serre fibration $F\overset{\subseteq}{\rightarrow}Y\overset{\pi}{\longrightarrow}B$ and a continuous map of topological spaces $f:X\rightarrow B$. We have the following diagram

$$\begin{array}{ccc} F & = & F \\ \downarrow & & \downarrow \\ X \times_B Y & \xrightarrow{f^*} & Y \\ {\scriptstyle \pi^*}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[f]{} & B \end{array}$$

This gives rise to the following diagram in cochains

$$\begin{array}{ccc} C^*(F;K) & = & C^*(F;K) \\ \uparrow & & \uparrow \\ C^*(X \times_B Y;K) & \xleftarrow{f^{*\#}} & C^*(Y;K) \\ {\scriptstyle \pi^{*\#}}\uparrow & & \uparrow{\scriptstyle \pi^{\#}} \\ C^*(X;K) & \xleftarrow[f^{\#}]{} & C^*(B;K) \end{array}$$

Therefore we can regard $C^*(X;K)$ as a right differential $C^*(B;K)$-module and $C^*(Y;K)$ as a left differential $C^*(B;K)$-module in the usual way. Thus $\mathrm{Tor}_{C^*(B;K)}(C^*(X;K),C^*(Y;K))$ is defined.

<u>THEOREM 2</u>: Given a Serre fibration $F \to Y \xrightarrow{\pi} B$ and a continuous map of topological spaces $f: X \to B$, there is a module isomorphism

$$\bar{\sigma}: \mathrm{Tor}_{C^*(B;K)}(C^*(X;K),C^*(Y;K)) \xrightarrow{\approx} H^*(X \times_B Y;K).$$

<u>REMARK</u>: For a proof of Theorem 2 see Eilenberg and Moore [11][12], Baum [1], or Smith [20]. We shall apply Theorem 2 to differentiable fibre bundles, and in particular to homogeneous spaces. So let

$$\sigma = (E, \pi, X, G/H, G)$$

be a differentiable fibre bundle, where G is a compact, connected Lie group and H a compact, connected subgroup of G, E and X are differentiable manifolds, and $\pi: E \to X$ is a differentiable map. We then have a universal bundle

$$\sigma(G,H) = (BH, f, BG, G/H, G)$$

and the following classifying diagram

$$\begin{array}{ccc} G/H & = & G/H \\ \downarrow & & \downarrow \\ E & \longrightarrow & BH \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[g]{} & BG \end{array}$$

In the special case $X = *$ is a point we are reduced to the following classifying diagram

$$\begin{array}{ccc} G/H & = & G/H \\ \| & & \downarrow \\ G/H & \longrightarrow & BH \\ \downarrow & & \downarrow{\scriptstyle f} \\ * & \longrightarrow & BG \end{array}$$

So now we have two corollaries of Theorem 2:

<u>COROLLARY 3</u>: Given a differentiable fibre bundle $\sigma = (E, \pi, X, G/H, G)$ there is a module isomorphism

$$\overline{\sigma}: \mathrm{Tor}_{C^*(BG;K)}(C^*(X;K), C^*(BH;K)) \xrightarrow{\approx} H^*(E;K).$$

<u>COROLLARY 4</u>: Given a homogeneous space G/H there is

a module isomorphism

$$\bar{\sigma}: \mathrm{Tor}_{C^*(BG;K)}(K, C^*(BH;K)) \xrightarrow{\approx} H^*(G/H;K).$$

REMARK: We shall make use of Theorem 2 in the following manner: By Theorem II.1.1 there exists an Eilenberg - Moore spectral sequence (E_r, d_r) such that

$$\mathrm{tor}_{H^*(B;K)}(H^*(X;K), H^*(Y;K)) = E_2 \Longrightarrow$$

$$E_\infty = \mathrm{Tor}_{C^*(B;K)}(C^*(X;K), C^*(Y;K)) \approx H^*(X \;_B Y;K).$$

In particular (Corollary 4), there exists an Eilenberg - Moore spectral sequence (E_r, d_r) such that

$$\mathrm{tor}_{H^*(BG;K)}(K, H^*(BH;K)) = E_2 \Longrightarrow$$

$$E_\infty = \mathrm{Tor}_{C^*(BG;K)}(K, C^*(BH;K)) \approx H^*(G/H;K).$$

Thus for the purposes of Theorem 1 we would like to prove that $E_2 = E_\infty$ in the Eilenberg - Moore spectral sequence for G/H. The following theorem is due to Baum [1][2], and may be interpreted as saying that it is sufficient to prove that $E_2 = E_\infty$ in the Eilenberg - Moore spectral sequence for G/T, T a maximal torus of H.

THEOREM 5: If G/H is a homogeneous space and T is a maximal torus of H, Then $E_2 = E_\infty$ in the Eilenberg - Moore spectral sequence for G/H if and only if $E_2 = E_\infty$ in the Eilenberg - Moore spectral sequence for G/T.

REMARK: To exploit this reduction to the case of a torus T, we use a result announced by May [16]. The theorem is based on work of H. Cartan [6], and is proved in explicit detail in the appendix of Gugenheim and May [13].

THEOREM 6: There exists a differential multiplicative map

$$\alpha: C^*(BT;K) \to H^*(BT;K)$$

which induces the identity in homology and annihilates $\cup_1$-products.

REMARK: We give a rough sketch of this proof; for further details see Gugenheim and May [13].

Recall that $BT = B(S^1 \times \ldots(n)\ldots \times S^1)$ is the Eilenberg - MacLane space $K(Z \oplus \ldots(n)\ldots \oplus Z, 2)$. (See, for example, H. Cartan [6] or Eilenberg and MacLane [10].) Write $\pi = Z \oplus \ldots(n)\ldots \oplus Z$. Then, letting $F\pi$ denote the group ring of π over K, write $\overline{B}^{(0)}(\pi) = F\pi$, and, inductively, $\overline{B}^{(n)}(\pi) = \overline{B}(\overline{B}^{(n-1)}(\pi))$. Recall the $\overline{W}$ construction (due originally to Eilenberg and MacLane [10]), which we may iterate analogously. By results of May we can replace the cochains and chains of BT by the cochains and chains of $W^{(2)}(\pi)$. Using this fact, we construct a differential comultiplicative map

$$\beta: \overline{B}^{(2)}(\pi) \to C_*(BT;K)$$

which respects the homotopy cocommutativity and induces the identity in homology.

Now $\overline{B}^{(1)}(\pi)$ and $\overline{B}^{(2)}(\pi)$ are homotopy cocommutative via maps

$$\cap_1^1 : \overline{B}^{(1)}(\pi) \to \overline{B}^{(1)}(\pi) \otimes \overline{B}^{(1)}(\pi),$$

$$\cap_1^2 : \overline{B}^{(2)}(\pi) \to \overline{B}^{(2)}(\pi) \otimes \overline{B}^{(2)}(\pi),$$

respectively. Also $\overline{B}H_*(K(\pi,1);K)$ is homotopy cocommutative via a map

$$\cap_1 : \overline{B}H_*(K(\pi,1);K) \to \overline{B}H_*(K(\pi,1);K) \otimes \overline{B}H_*(K(\pi,1);K).$$

These homotopies are defined inductively and satisfy certain naturality properties.

Next, utilizing the little constructions $\overline{K}$ of H. Cartan [6], we construct by induction differential comultiplicative maps

$$\gamma^1 : H_*(K(\pi,1);K) = \overline{K}F\pi \to \overline{B}F\pi = \overline{B}^{(1)}(\pi),$$

$$\gamma^2 : H_*(BT;K) = H_*(K(\pi,2);K) = \overline{K}H_*(K(\pi,1);K) \to \overline{B}H_*(K(\pi,1);K)$$

which induce the identity in homology.

By a fairly straightforward inductive argument, May shows that

$$\cap_1 \gamma^2 \equiv 0.$$

Next, he defines

$$\delta : H_*(BT;K) \to \overline{B}H_*(K(\pi,1);K) \to \overline{B}(\overline{B}^{(1)}(\pi)) = \overline{B}^{(2)}(\pi)$$

as the composition

$$\delta := \overline{B}(\gamma^1) \circ \gamma^2.$$

Then by naturality it follows that

$$\cap^2_1\delta = \cap^2_1\overline{B}(\gamma^1)\gamma^2 = (\overline{B}(\gamma^1)\otimes\overline{B}(\gamma^1))\cap_1\gamma^2 = 0.$$

Finally, we let α be the dual of the composition

$$\beta\circ\delta : H_*(BT;K)\to C_*(BT;K).$$

Clearly α satisfies the conditions of Theorem 6.

<u>PROOF OF THEOREM 1</u>: By Theorem 5 we are reduced to proving Theorem 1 in the special case where the subgroup of G is a torus T. We have the following classifying diagram:

$$\begin{array}{ccc}
G/T & = & G/T \\
\| & & \downarrow \\
G/T & \longrightarrow & BT \\
\downarrow & & \downarrow f \\
* & \longrightarrow & BG
\end{array}$$

This gives rise to the following diagram in cochains:

$$\begin{array}{ccc}
C^*(G/T;K) & = & C^*(G/T;K) \\
\| & & \uparrow \\
C^*(G/T;K) & \longleftarrow & C^*(BT;K) \\
\uparrow & & \uparrow f^{\#} \\
K & \longleftarrow & C^*(BG;K)
\end{array}$$

It also gives rise to the following diagram in cohomology:

$$
\begin{array}{ccc}
H^*(G/T;K) & = & H^*(G/T;K) \\
\| & & \uparrow \\
H^*(G/T;K) & \longleftarrow & H^*(BT;K) \\
\uparrow & & \uparrow f^* \\
K & \longleftarrow & H^*(BG;K)
\end{array}
$$

Next consider the following diagram:

$$
(*) \qquad
\begin{array}{ccccc}
C^*(BT;K) & \xleftarrow{f^\#} & C^*(BG;K) & \longrightarrow & K \\
\alpha \updownarrow & & \uparrow \theta_1 & & \| \\
H^*(BT;K) & \xleftarrow[f^*]{} & H^*(BG;K) & \longrightarrow & K
\end{array}
$$

where

(i). α is the map given by Theorem 6;

(ii). θ_1 is the first term of the shm map $\{\theta_1, \theta_2, \theta_3, \ldots\}$ from $H^*(BG;K)$ to $C^*(BG;K)$ given by Theorem II.2.3.

Now the preceeding diagram certainly commutes in homology, since $(\alpha f^\# \theta_1)_* = f^*$. But $H^*(BT;K)$ and $H^*(BG;K)$ have 0 differentials. In other words, (*) actually commutes. So

$$\alpha f^\# \theta_1 = f^*.$$

Also, unless $m = 1$, the composition

$$\alpha f^\# \theta_m : H^*(BG;K) \otimes \ldots (m) \ldots \otimes H^*(BG;K) \to H^*(BT;K)$$

is identically 0, because each term of θ_m contains at least one $\cup_1$-product, $f^\#$ commutes with $\cup_1$-products, and α annihilates $\cup_1$-products.

These remarks imply that the shm map

$$(\{\alpha f^{\#}, 0, 0, 0, \ldots\})\circ(\{\theta_1, \theta_2, \theta_3, \ldots\})$$

is identically equal to the (actually strictly multiplicative) shm map

$$\{f^*, 0, 0, 0, \ldots\}.$$

Utilizing Corollary 4, Corollary II.1.3, and Corollary II.1.11 we now have a string of module isomorphisms...

$$\begin{array}{c} H^*(G/T;K) \\ \uparrow \approx \; \overline{\sigma} \\ \mathrm{Tor}_{C^*(BG;K)}(K, C^*(BT;K)) \\ \downarrow \approx \; \mathrm{Tor}_1(1,\alpha) \\ \mathrm{Tor}_{C^*(BG;K)}(K, H^*(BT;K)) \\ \uparrow \approx \; \mathrm{TOR}_{\theta_*}(1,1) \\ \mathrm{TOR}_{H^*(BG;K)}(K, H^*(BT;K)) \\ \updownarrow = \\ \mathrm{tor}_{H^*(BG;K)}(K, H^*(BT;K)). \end{array}$$

This completes the proof of Theorem 1.

REMARK: One might conjecture (as Hirsch did) the existence of a module isomorphism

$$H^*(G/H;K) \approx \mathrm{tor}_{H^*(BG;K)}(K, H^*(BH;K))$$

along the lines of Theorem 1 in total generality. However, Schochet [19] has given a counterexample.

III. THE REAL AND RATIONAL COHOMOLOGY OF DIFFERENTIABLE FIBRE BUNDLES

III. 1. tor, Tor AND THE TWO-SIDED KOSZUL CONSTRUCTION:

The major theorem in this chapter expresses, under certain reasonable hypotheses, the real or rational cohomology of a differentiable fibre bundle as a certain torsion product. Whereas in II we were only able to obtain module isomorphisms, in this chapter we will obtain actual algebra isomorphisms; so let us first describe the algebra structure involved.

First of all, suppose that A is a differential graded algebra over K, M is a right differential A-module, and N is a left differential A-module. As usual there is a natural map

$$e:(M\otimes\overline{B}(A)\otimes N)\otimes(M\otimes\overline{B}(A)\otimes N)\to M\otimes M\otimes\overline{B}(A\otimes A)\otimes N\otimes N$$

which, on passing to homology, gives a natural map

$$e^*:\mathrm{Tor}_A(M,N)\otimes\mathrm{Tor}_A(M,N)\to\mathrm{Tor}_{A\otimes A}(M\otimes M,N\otimes N).$$

e^* is called the EXTERNAL PRODUCT.

Now suppose that A, M, and N are graded commutative differential graded algebras over K with multiplication maps $\triangle, \triangle_1, \triangle_2$, respectively. If M is regarded as a right differential A-module via a differential multiplicative map $\alpha:A\to M$ and N is regarded as a left differential A-module via a differential multiplicative map $\beta:A\to N$ then

$$\triangle\ :A\otimes A\to A,$$

$$\triangle_1:M\otimes M\to M,$$

and

$$\Delta_2 : N \otimes N \to N$$

are differential multiplicative maps. Define

$$\mathrm{Tor}_{\Delta}(\Delta_1, \Delta_2) : \mathrm{Tor}_{A \otimes A}(M \otimes M, N \otimes N) \to \mathrm{Tor}_A(M,N)$$

as the composition

$$\mathrm{Tor}_{\Delta}(1,1) \circ \mathrm{Tor}_1(\Delta_1, 1) \circ \mathrm{Tor}_1(1, \Delta_2).$$

Finally, define an algebra structure on $\mathrm{Tor}_A(M,N)$ by the composition

$$\mathrm{Tor}_{\Delta}(\Delta_1, \Delta_2) \circ e^* : \mathrm{Tor}_A(M,N) \otimes \mathrm{Tor}_A(M,N) \to \mathrm{Tor}_A(M,N).$$

Then, under the additional assumption of graded commutativity, it is not difficult to show that the isomorphisms

$$\mathrm{Tor}_g(1,1) : \mathrm{Tor}_{A_2}(M_1,N_1) \xrightarrow{\approx} \mathrm{Tor}_{A_1}(M_1,N_1),$$

$$\mathrm{Tor}_1(1,f) : \mathrm{Tor}_{A_2}(M_2,N_2) \xrightarrow{\approx} \mathrm{Tor}_{A_2}(M_2,N_1),$$

$$\mathrm{Tor}_1(h,1) : \mathrm{Tor}_{A_2}(M_2,N_2) \xrightarrow{\approx} \mathrm{Tor}_{A_2}(M_1,N_2)$$

of Corollary II.1.3, and the isomorphisms

$$\overline{\sigma} : \mathrm{Tor}_{C^*(B;K)}(C^*(X;K), C^*(Y;K)) \xrightarrow{\approx} H^*(X \;_B Y;K),$$

$$\overline{\sigma} : \mathrm{Tor}_{C^*(BG;K)}(C^*(X;K), C^*(BH;K)) \xrightarrow{\approx} H^*(E;K),$$

$$\overline{\sigma} : \mathrm{Tor}_{C^*(BG;K)}(K, C^*(BH;K)) \xrightarrow{\approx} H^*(G/H;K),$$

of Theorem II.3.2 and Corollaries II.3.3 and II.3.4, respectively, are indeed algebra isomorphisms.

The torsion products $tor_A(M,N)$ and $Tor_A(M,N)$ will be redefined in this section, in the special case where

$$A = P[x_1,\dots,x_n]$$

is a polynomial algebra, in terms of some form of the two-sided Koszul construction. It is worth noting that in each case the two-sided Koszul construction could also be described by defining the so-called Koszul constrution, tensoring on the two sides, and noting that the additional structure, namely the differential, is induced naturally from the various components.

For the remainder of this chapter fix K to be a field. The following material is valid in a somewhat more general context but this will not be needed. Suppose $P[x_1,\dots,x_n]$ is a polynomial algebra over K. Consider the exterior algebra

$$E[u_1,\dots,u_n]$$

over K, where u_i has INTERNAL degree $Deg(x_i)$, EXTERNAL degree -1, bidegree $(Deg(x_i),-1)$, and hence degree $Deg(x_i)-1$ in the associated graded algebra over K.

(a). <u>tor</u>:

Suppose that M is a right $P[x_1,\dots,x_n]$-module and N is a left $P[x_1,\dots,x_n]$-module. We form the complex $M\otimes E[u_1,\dots,u_n]\otimes N$ with the natural differential d_E given by

$$d_E(m \otimes 1 \otimes n) = 0,$$

$$d_E(m \otimes u_i \otimes n) = mx_i \otimes 1 \otimes n + m \otimes 1 \otimes x_i n,$$

$$d_E \text{ a derivation.}$$

d_E is called the EXTERNAL differential, since it acts on external degree. We will call the complex

$$(M \otimes E[u_1, \ldots, u_n] \otimes N, d_E)$$

the FIRST TWO-SIDED KOSZUL CONSTRUCTION. Observe that the composition $d_E \circ d_E \equiv 0$. The first two-sided Koszul construction thus has the structure of a differential graded module over K.

THEOREM 1: $\operatorname{tor}_{P[x_1,\ldots,x_n]}(M,N)$ is the homology of the first two-sided Koszul construction:

$$\operatorname{tor}_{P[x_1,\ldots,x_n]}(M,N) \approx H(M \otimes E[u_1,\ldots,u_n] \otimes N, d_E).$$

REMARK: To prove Theorem 1 one simply checks that the first two-sided Koszul construction is a projective resolution. See, for example, Baum and Smith [3].

(b). Tor:

Now suppose that M is a right differential $P[x_1,\ldots,x_n]$-module and N is a left differential $P[x_1,\ldots,x_n]$-module. We again form the complex $M \otimes E[u_1,\ldots,u_n] \otimes N$, this time with the natural differential $d_E = d_E + d_I$, where

$$d_I(m \otimes 1 \otimes n) = dm \otimes 1 \otimes n + (-1)^{\operatorname{Deg}(m)} m \otimes 1 \otimes dn,$$

$$d_I(m \otimes u_i \otimes n) = -\,dm \otimes u_i \otimes n - (-1)^{\mathrm{Deg}(m)} m \otimes u_i \otimes dn,$$

d_I a derivation.

d_I is called the INTERNAL differential, since it acts on internal degree. We will call the complex

$$(M \otimes E[u_1, \ldots, u_n] \otimes N, d_D)$$

the SECOND TWO-SIDED KOSZUL CONSTRUCTION. Observe that the signs have been chosen so that $d_D \circ d_D \equiv 0$. The second two-sided Koszul construction thus has the structure of a differntial graded module over K.

THEOREM 2: $\mathrm{Tor}_{P[x_1,\ldots,x_n]}(M,N)$ is the homology of the second two-sided Koszul construction:

$$\mathrm{Tor}_{P[x_1,\ldots,x_n]}(M,N) \approx H(M \otimes E[u_1,\ldots,u_n] \otimes N, d_I).$$

REMARK: To prove Theorem 2 one simply checks that the second two-sided Koszul construction is a differential projective resolution. See, for example, Baum and Smith [3].

REMARK: We now prove the analog of the comparison Theorem II.1.4 (and Theorem II.1.12). We observe that historically the order should be reversed: Theorem 3 below is used in Baum [1] and Baum and Smith [3]; Theorems II.1.4 and II.1.12 were based philosophically on Theorem 3.

THEOREM 3: Suppose $P[x_1,\ldots,x_n]$ is a polynomial algebra over K, M and N are differential graded algebras over K, and $f,g: P[x_1,\ldots,x_n] \to N$ and $h: P[x_1,\ldots,x_n] \to M$ are differential multiplicative maps. If f and g are chain homotopic, then $Tor_{P[x_1,\ldots,x_n]}(M,N)$ is unambiguously defined; that is, $Tor_{P[x_1,\ldots,x_n]}(M,N)$ is the same whether N is regarded as a left differential $P[x_1,\ldots,x_n]$-module via f or via g:

$$(Tor_{P[x_1,\ldots,x_n]}(M,N))_f \approx (Tor_{P[x_1,\ldots,x_n]}(M,N))_g.$$

An analogous result is true for $Tor_{P[x_1,\ldots,x_n]}(N,M)$:

$$(Tor_{P[x_1,\ldots,x_n]}(N,M))_f \approx (Tor_{P[x_1,\ldots,x_n]}(N,M))_g.$$

PROOF: We form $M \otimes E[u_1,\ldots,u_n] \otimes N$ with the differential d_f obtained via f and with the differential d_g obtained via g. Now construct the map

$$T: (M \otimes E[u_1,\ldots,u_n] \otimes N, d_f) \to (M \otimes E[u_1,\ldots,u_n] \otimes N, d_g)$$

as follows: Since f and g are chain homotopic, there exists, for each $i = 1,\ldots,n$, an element $h_i \in N$ such that

$$f(x_i) = g(x_i) - d(h_i).$$

Therefore set

$$T(m \otimes 1 \otimes n) = m \otimes 1 \otimes n,$$

$$T(1 \otimes u_i \otimes 1) = 1 \otimes u_i \otimes 1 - 1 \otimes 1 \otimes h_i.$$

We claim that T is a map of differential graded algebras; the proof is a direct calculation:

(i). $Td_f(m \otimes 1 \otimes n) = dm \otimes 1 \otimes n + (-1)^{Deg(m)} m \otimes 1 \otimes dn =$

$= d_g T(m \otimes 1 \otimes n);$

(ii). $Td_f(1\otimes u_i\otimes 1) = h(x_i)\otimes 1\otimes 1 + 1\otimes 1\otimes f(x_i) =$

$= h(x_i)\otimes 1\otimes 1 + 1\otimes 1\otimes g(x_i) - 1\otimes 1\otimes d(h_i) =$

$= d_g(1\otimes u_i\otimes 1 - 1\otimes 1\otimes h_i) = d_g T(1\otimes u_i\otimes 1).$

Since T has an obvious inverse it follows that T induces a module isomorphism

$$T_*:(\mathrm{Tor}_{P[x_1,\ldots,x_n]}(M,N))_f \xrightarrow{\approx} (\mathrm{Tor}_{P[x_1,\ldots,x_n]}(M,N))_g.$$

The second assertion of Theorem 3 is proved analogously.

REMARK: Under the additional assumption that M and N are graded commutative it is easy to see that the isomorphisms

$$T_*:(\mathrm{Tor}_{P[x_1,\ldots,x_n]}(M,N))_f \xrightarrow{\approx} (\mathrm{Tor}_{P[x_1,\ldots,x_n]}(M,N))_g,$$

$$T_*:(\mathrm{Tor}_{P[x_1,\ldots,x_n]}(N,M))_f \xrightarrow{\approx} (\mathrm{Tor}_{P[x_1,\ldots,x_n]}(N,M))_g$$

are indeed algebra isomorphisms.

III. 2. THE REAL AND RATIONAL COHOMOLOGY OF DIFFERENTIABLE FIBRE BUNDLES:

It will be convenient to present our results in real coefficients first. A trivial remark at the end of this section will extend each theorem to rational coefficients as well.

It will be instructive to begin by giving the proof of the homogeneous space theorem with real coefficients. The techniques used in this proof served as a philosophical base for the results in II; they will also serve as an introduction to the types of arguments used in this chapter. The proof of this theorem is due to Baum [1].

THEOREM 1: Let G be a compact, connected Lie group and H a compact, connected subgroup of F. Form the homogeneous space G/H. Then, regarding **R** as a right $H^*(BG;\mathbf{R})$-module via augmentation and $H^*(BH;\mathbf{R})$ as a left $H^*(BG;\mathbf{R})$-module via the natural map $f^*: H^*(BG;\mathbf{R}) \to H^*(BH;\mathbf{R})$, we have an algebra isomorphism

$$H^*(G/H;\mathbf{R}) \approx \mathrm{tor}_{H^*(BG;\mathbf{R})}(\mathbf{R}, H^*(BH;\mathbf{R})).$$

PROOF: We fix the following notation: If M is a Riemannian manifold modeled on a separable Hilbert space, then we denote by $R^{\#}(M,d)$ the differential graded algebra of deRham cochains with exterior derivative. Recall that we have a natural algebra isomorphism

$$H^*(M;\mathbf{R}) \approx H(R^{\#}(M,d)).$$

We have the following classifying diagram:

$$\begin{array}{ccc} G/H & = & G/H \\ \| & & \downarrow \\ G/H & \longrightarrow & BH \\ \downarrow & & \downarrow f \\ * & \longrightarrow & BG \end{array}$$

This gives rise to the following diagram in deRham cochains:

$$\begin{array}{ccc} R^{\#}(G/H) & = & R^{\#}(G/H) \\ \| & & \uparrow \\ R^{\#}(G/H) & \longleftarrow & R^{\#}(BH) \\ \uparrow & & \uparrow f^{\#} \\ R & \longleftarrow & R^{\#}(BG) \end{array}$$

It also gives rise to the following diagram in cohomology:

$$\begin{array}{ccc} H^*(G/H;R) & = & H^*(G/H;R) \\ \| & & \uparrow \\ H^*(G/H;R) & \longleftarrow & H^*(BH;R) \\ \uparrow & & \uparrow f^* \\ R & \longleftarrow & H^*(BG;R) \end{array}$$

We may assume that BH and BG are differential manifolds modeled on separable Hilbert spaces and that all the maps in the classifying diagram above are differentiable.

We know that $H^*(BG;R)$ and $H^*(BH;R)$ are polynomial algebras on generators of even degree. In fact, let

$$H^*(BG;R) = P[x_1,\dots,x_m],$$

$$H^*(BH;R) = P[y_1,\ldots,y_n]$$

Now choose arbitrary representative cocycles $u_1,\ldots,u_m$ in $R^{\#}(BG)$ for $x_1,\ldots,x_m$. We define the map θ as follows: For each $i = 1,\ldots,m$, define $\theta(x_i) = u_i$. Since $R^{\#}(BG)$ is graded commutative the map extends to a unique differential multiplicative map

$$\theta: H^*(BG;R) \to R^{\#}(BG).$$

From its definition it is clear that θ induces the identity map in homology. (Compare II. 2 !)

Similarly we construct a differential multiplicative map

$$\emptyset: H^*(BH;R) \to R^{\#}(BH)$$

which also induces the identity map in homology.

Next consider the following diagram (which we do not claim to be commutative):

(*)
$$\begin{array}{ccccc} R^{\#}(BH) & \xleftarrow{f^{\#}} & R^{\#}(BG) & \longrightarrow & R \\ \emptyset \uparrow & & \theta \uparrow & & \parallel \\ H^*(BH;R) & \xleftarrow[f^*]{} & H^*(BG;R) & \longrightarrow & R \end{array}$$

Observe that $R^{\#}(BH)$ can be regarded as a left differential $H^*(BG;R)$-module in two distinct ways; via the differential multiplicative map $f^{\#}\theta$ or via the differential multiplicative map $\emptyset f^*$. This gives rise to two distinct torsion products, which we shall denote, respectively, by

$$^{\urcorner}\mathrm{Tor}_{H^*(BG;R)}(R, R^{\#}(BH))$$

and by

$$\uparrow\!\!\llcorner\, \mathrm{Tor}_{H^*(BG;R)}(R,R^{\#}(BH)).$$

Now the preceeding diagram certainly commutes in homology, since $(f^{\#}\theta)_* = f^* = (\emptyset f^*)_*$. Therefore $f^{\#}\theta$ and $\emptyset f^*$ are chain homotopic. Hence Theorem III.1.3 applies.

Utilizing Corollary II.3.4, Corollary II.1.3, and Theorem III.1.3, we now have a string of algebra isomorphisms...

$$\begin{array}{c}
H^*(G/H;R) \\
\approx \Big\uparrow \bar{\sigma} \\
\mathrm{Tor}_{R^{\#}(BG)}(R,R^{\#}(BH)) \\
\approx \Big\uparrow \mathrm{Tor}_{\theta}(1,1) \\
\leftarrow\!\!\urcorner\mathrm{Tor}_{H^*(BG;R)}(R,R^{\#}(BH)) \\
\approx \Big\downarrow T_* \\
\uparrow\!\!\llcorner\, \mathrm{Tor}_{H^*(BG;R)}(R,R^{\#}(BH)) \\
\approx \Big\uparrow \mathrm{Tor}_1(1,\emptyset) \\
\mathrm{tor}_{H^*(BG;R)}(R,H^*(BH;R)).
\end{array}$$

This completes the proof of Theorem 1.

<u>REMARK</u>: We now try to extend the type of reasoning involved in Theorem 1 to differentiable fibre bundles. So let

$$\sigma = (E,\pi,X,G/H,G)$$

be a differentiable fibre bundle, where G is a compact, connected Lie group and H a compact, connected subgroup of G, E and X are differentiable manifolds, and $\pi: E \to X$

is a differentiable map.

We then have a universal bundle

$$\sigma(G,H) = (BH,f,BG,G/H,G)$$

and the following classifying diagram:

$$\begin{array}{ccc}
G/H & = & G/H \\
\downarrow & & \downarrow \\
E & \longrightarrow & BH \\
\pi\downarrow & & \downarrow f \\
X & \xrightarrow{\;g\;} & BG
\end{array}$$

This gives rise to the following diagram in deRham cochains:

$$\begin{array}{ccc}
R^{\#}(G/H) & = & R^{\#}(G/H) \\
\uparrow & & \uparrow \\
R^{\#}(E) & \longleftarrow & R^{\#}(BH) \\
\pi^{\#}\uparrow & & \uparrow f^{\#} \\
R^{\#}(X) & \xleftarrow{\;g^{\#}\;} & R^{\#}(BG)
\end{array}$$

It also gives rise to the following diagram in cohomology:

$$\begin{array}{ccc}
H^*(G/H;R) & = & H^*(G/H;R) \\
\uparrow & & \uparrow \\
H^*(E;R) & \longleftarrow & H^*(BH;R) \\
\pi^*\uparrow & & \uparrow f^* \\
H^*(X;R) & \xleftarrow{\;g^*\;} & H^*(BG;R)
\end{array}$$

Now of course the theorem we desire says: Under "reasonable" hypotheses on the differentiable manifold X, by regarding

$H^*(X;R)$ as a right $H^*(BG;R)$-module via the map g^* and $H^*(BH;R)$ as a left $H^*(BG;R)$-module via the map f^*, we have an algebra isomorphism

$$H^*(E;R) \approx \mathrm{tor}_{H^*(BG;R)}(H^*(X;R),H^*(BH;R)).$$

Our goal is to see how far we can relax the conditions on X and still obtain this isomorphism. The following two corollaries of Theorem 1 and its proof are obvious:

<u>COROLLARY 2</u>: If

$$\sigma = (E,\pi,X,G/H,G)$$

is a differentiable fibre bundle with X a Riemannian symmetric space, then there is an algebra isomorphism

$$H^*(E;R) \approx \mathrm{tor}_{H^*(BG;R)}(H^*(X;R),H^*(BH;R)).$$

<u>COROLLARY 3</u>: If

$$\sigma = (E,\pi,X,G/H,G)$$

is a differentiable fibre bundle with $H^*(X;R)$ a polynomial algebra, then there is an algebra isomorphism

$$H^*(E;R) \approx \mathrm{tor}_{H^*(BG;R)}(H^*(X;R),H^*(BH;R)).$$

<u>REMARK</u>: Corollary 2 is a consequence of the following very special property of a Riemannian symmetric space X: The product $b_1 \wedge b_2$ of two harmonic forms $b_1, b_2 \in R^{\#}(X)$ is again a harmonic form. Therefore the map

$$\alpha: H^*(X;R) \to R^{\#}(X)$$

defined by sending each element $x \in H^*(X;R)$ into the unique harmonic form $\alpha(x)$ whose class is x is a differential multiplicative map which clearly induces the identity in homology. Corollary 2 is due to Baum and Smith [3].

<u>REMARK</u>: The condition of Corollary 3 is satisfied, for example, if X is itself the classifying space of a compact connected Lie group.

<u>THEOREM 4</u>: If

$$\sigma = (E, \pi, X, G/H, G)$$

is a differentiable fibre bundle with X a homogeneous space formed as the quotient G'/H' of a compact, connected Lie group G' by a compact, connected subgroup H' of maximal rank in G', then there is an algebra isomorphism

$$H^*(E;R) \approx \mathrm{tor}_{H^*(BG;R)}(H^*(X;R), H^*(BH;R)).$$

<u>PROOF</u>: We have the following classifying diagram:

$$\begin{array}{ccc} G'/H' & = & G'/H' \\ \| & & \downarrow \\ G'/H' & \xrightarrow{k} & BH' \\ \downarrow & & \downarrow h \\ * & \longrightarrow & BG' \end{array}$$

This gives rise to the following diagram in deRham cochains:

$$\begin{array}{ccc}
R^{\#}(G'/H') & = & R^{\#}(G'/H') \\
\| & & \uparrow \\
R^{\#}(G'/H') & \xleftarrow{k^{\#}} & R^{\#}(BH') \\
\uparrow & & \uparrow h^{\#} \\
R & \longleftarrow & R^{\#}(BG')
\end{array}$$

It also gives rise to the following diagram in cohomology:

$$\begin{array}{ccc}
H^*(G'/H';R) & = & H^*(G'/H';R) \\
\| & & \uparrow \\
H^*(G'/H';R) & \xleftarrow{k^*} & H^*(BH';R) \\
\uparrow & & \uparrow h^* \\
R & \longleftarrow & H^*(BG';R)
\end{array}$$

We recall now the relevant facts about maximal rank spaces: The fact that

$$H^*(BG';R) = P[x_1,\ldots,x_m],$$

$$H^*(BH';R) = P[y_1,\ldots,y_n]$$

are polynomial algebras on generators of even degree is equivalent to the fact that

$$H^*(G';R) = E[u_1,\ldots,u_m],$$

$$H^*(H';R) = E[v_1,\ldots,v_n]$$

are exterior algebras on generators of odd degree. RANK is the number of generators: Rank(G') = m; Rank(H') = n. H' has MAXIMAL RANK in G' if m = n. It then follows that (i). The sequence

$$R \to H^*(BG';R) \xrightarrow{h^*} H^*(BH';R) \xrightarrow{k^*} H^*(G'/H';R) \to R$$

is co-exact;

(ii). As an $H^*(BG';R)$-module, $H^*(BH';R)$ is isomorphic to $H^*(BG';R) \otimes H^*(G'/H';R)$.

For further details see Baum [1][2].

Now construct differential multiplicative maps

$$\gamma : H^*(BH';R) \to R^{\#}(BH'),$$

$$\delta : H^*(BG';R) \to R^{\#}(BG')$$

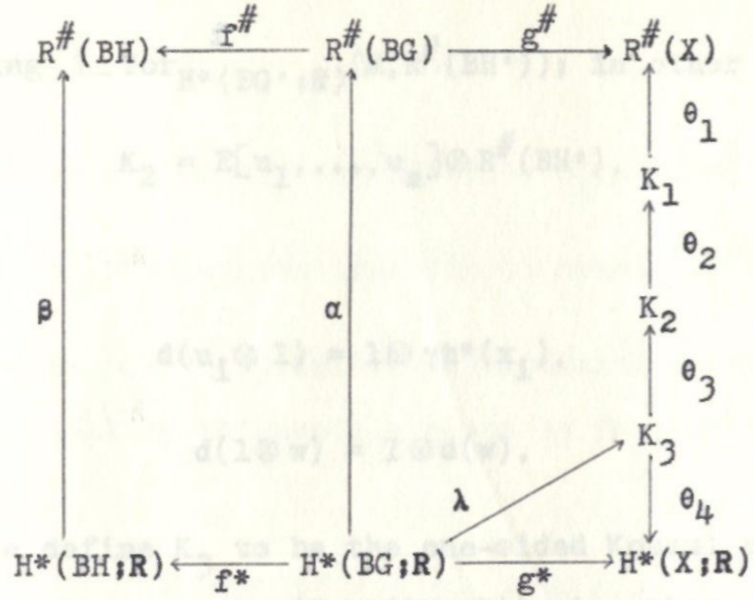
which induce the identity in homology, by analogy with the corresponding maps constructed in the proof of Theorem 1.

Next consider the following diagram (which we do not claim to be commutative):

$$(*) \qquad \begin{array}{ccccc} R^{\#}(BH') & \xleftarrow{h^{\#}} & R^{\#}(BG') & \longrightarrow & R \\ \uparrow \gamma & & \uparrow \delta & & \| \\ H^*(BH';R) & \xleftarrow[h^*]{} & H^*(BG';R) & \longrightarrow & R \end{array}$$

Using the above we are finally able to consider the following diagram (which, again, we do not claim to be commutative):

(**)



Various complexes and maps have not yet been defined, and we proceed as follows:

(i). We construct differential multiplicative maps

$$\alpha : H^*(BG;R) \to R^{\#}(BG),$$

$$\beta : H^*(BH;R) \to R^{\#}(BH)$$

which induce the identity in homology, by analogy with the corresponding maps γ, δ above.

(ii). We define K_1 to be the one-sided Koszul construction for computing $\overleftarrow{}\!\!\!\mathrm{Tor}_{H^*(BG';R)}(R, R^{\#}(BH'))$; in other words,

$$K_1 = E[u_1, \ldots, u_m] \otimes R^{\#}(BH'),$$

where

$$d(u_i \otimes 1) = 1 \otimes h^{\#}\delta(x_i),$$

$$d(1 \otimes w) = 1 \otimes d(w).$$

(iii). We define K_2 to be the one-sided Koszul construction for computing $\underline{\uparrow}\operatorname{Tor}_{H^*(BG';R)}(R,R^{\#}(BH'))$; in other words

$$K_2 = E[u_1,\ldots,u_m]\otimes R^{\#}(BH'),$$

where

$$d(u_i\otimes 1) = 1\otimes \gamma h^*(x_i),$$

$$d(1\otimes w) = 1\otimes d(w).$$

(iv). We define K_3 to be the one-sided Koszul construction for computing $\operatorname{tor}_{H^*(BG';R)}(R,H^*(BH';R))$; in other words

$$K_3 = E[u_1,\ldots,u_m]\otimes H^*(BH';R),$$

where

$$d(u_i\otimes 1) = 1\otimes h^*(x_i),$$

$$d(1\otimes w) = 0.$$

(v). θ_1 is defined as follows: To define $\theta_1(u_i\otimes 1)$, we note that $k^{\#}h^{\#}\delta(x_i)$ is a coboundary in $R^{\#}(X)$. Therefore choose, for each $i = 1,\ldots,m$, an arbitrary element $r_i \in R^{\#}(X)$ such that $d(r_i) = k^{\#}h^{\#}\delta(x_i)$. Now set

$$\theta_1(u_i\otimes 1) = r_i,$$

$$\theta_1(1\otimes w) = k^{\#}(w).$$

The proof that θ_1 is a differential multiplicative map is a direct calculation:

(a). $d\theta_1(u_i\otimes 1) = d(r_i) = k^{\#}h^{\#}\delta(x_i) = \theta_1(1\otimes h^{\#}\delta(x_i)) =$

$= \theta_1 d(u_i \otimes 1)$.

(b). $d\theta_1(1 \otimes w) = d(k^{\#}(w)) = k^{\#}(d(w)) = \theta_1(1 \otimes d(w)) = \theta_1 d(1 \otimes w)$.

Observe that θ_1 induces the identity in homology.

(vi). θ_2 is the differential multiplicative map which induces the identity in homology, given by Theorem III.1.3; in other words

$$\theta_2(u_i \otimes 1) = u_i \otimes 1 - 1 \otimes s_i,$$

$$\theta_2(1 \otimes w) = 1 \otimes w,$$

where $s_i \in R^{\#}(BH')$ is such that $\gamma h^*(x_i) = h^{\#}\delta(x_i) - d(s_i)$.

(vii). θ_3 is defined as follows:

$$\theta_3(u_i \otimes 1) = u_i \otimes 1,$$

$$\theta_3(1 \otimes w) = 1 \otimes \gamma(w).$$

The proof that θ_3 is a differential multiplicative map is a direct calculation:

(a). $d\theta_3(u_i \otimes 1) = d(u_i \otimes 1) = 1 \otimes \gamma h^*(x_i) = \theta_3(1 \otimes h^*(x_i)) =$

$= \theta_3 d(u_i \otimes 1)$.

(b). $d\theta_3(1 \otimes w) = d(1 \otimes \gamma(w)) = 1 \otimes d(\gamma(w)) = 1 \otimes \gamma(d(w)) =$

$= 0 = \theta_3(0) = \theta_3 d(1 \otimes w)$.

Observe that θ_3 induces the identity in homology.

(viii). θ_4 is defined as follows:

$$\theta_4(u_i \otimes 1) = 0,$$

$$\theta_4(1 \otimes w) = k^*(w).$$

The proof that θ_4 is a differential multiplicative map is a direct calculation:

(a). $d\theta_4(u_i \otimes 1) = d(0) = 0 = k^*h^*(x_i) = \theta_4(1 \otimes h^*(x_i)) =$

$\theta_4 d(u_i \otimes 1)$.

(b). $d\theta_4(1 \otimes w) = d(k^*(w)) = 0 = \theta_4(0) = \theta_4 d(1 \otimes w)$.

(ix). We construct a differential multiplicative map

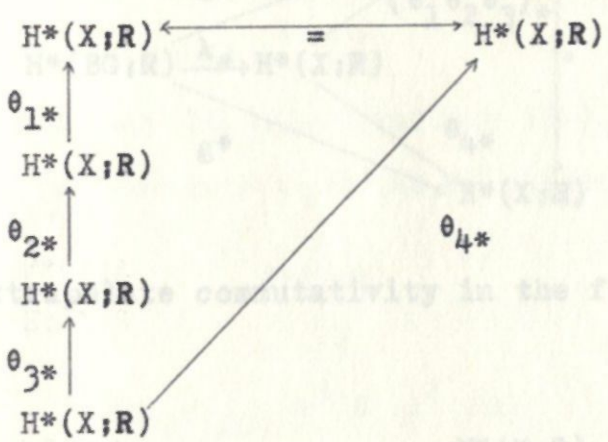
$$\lambda : H^*(BG;R) \to K_3$$

which induces g^* in homology, essentially by analogy with the maps α, β, γ, δ above: Let

$$H^*(BG;R) = P[z_1, \ldots, z_p]$$

be a polynomial algebra on generators of even degree. Consider $g^*(z_1), \ldots, g^*(z_p) \in H^*(X;R)$. Since $H(K_3, d) \approx$ $\approx H^*(X;R)$, we may choose arbitrary cocycles $t_1, \ldots, t_p \in K_3$ for $g^*(z_1), \ldots, g^*(z_p)$. For each $i = 1, \ldots, p$, define $\lambda(z_i) = t_i$. Since K_3 is graded commutative the map extends to a unique map λ satisfying the conditions above.

Given this diagram we consider the extreme right-hand side and claim that $\theta_1\theta_2\theta_3$ and θ_4 induce the same map in homology. In other words, we have commutativity in the following diagram:



To see this we examine the effect of applying the maps $\theta_1\theta_2\theta_3$ and θ_4 to a cycle in K_3. A cycle in K_3 has the form $1\otimes w$, where $dw = 0$. Now

(i). $\theta_1\theta_2\theta_3(1\otimes w) = \theta_1\theta_2(1\otimes \gamma(w)) = \theta_1(1\otimes \gamma(w)) =$

$= k^{\#}\gamma(w)$;

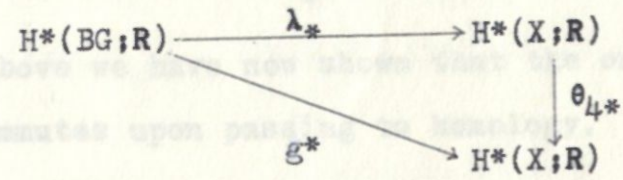
on the other hand

(ii). $\theta_4(1\otimes w) = k^*(w)$.

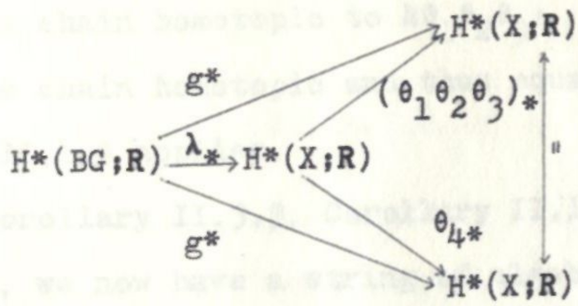
Thus $(\theta_1\theta_2\theta_3)_*([1\otimes w]) = (k^{\#}\gamma)_*([w]) = k^*([w]) = \theta_{4*}([1\otimes w])$

So the diagram commutes.

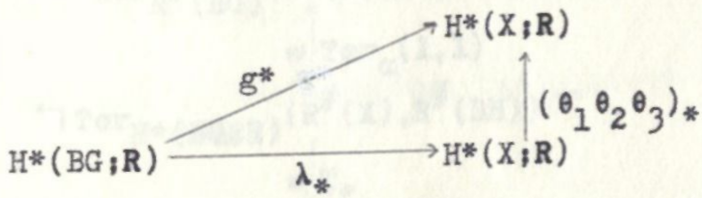
By the definition of λ we know that the following diagram is also commutative:



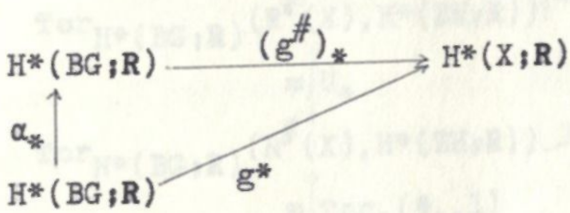
Thus the following diagram commutes also:



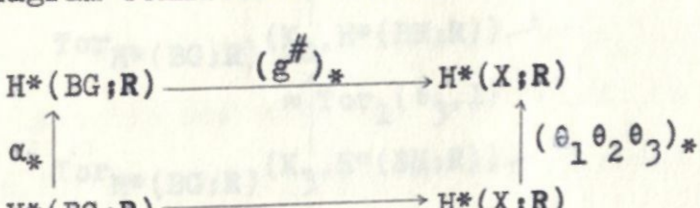
From this we extrapolate commutativity in the following diagram:



Since α_* is the identity and $(g^{\#})_* = g^*$, we have commutativity also in the following diagram:



Piecing together the two preceeding diagrams it follows that the next diagram commutes as well:



By all of the above we have now shown that the original diagram (**) commutes upon passing to homology.

Thus

(i). $f^{\#}\alpha$ is chain homotopic to βf^*;

(ii). $g^{\#}\alpha$ is chain homotopic to $\lambda\theta_1\theta_2\theta_3$;

(iii). $\lambda\theta_4$ is chain homotopic and thus equal to g^*.

Hence Theorem III.1.3 applies.

Utilizing Corollary II.3.3, Corollary II.1.3, and Theorem III.1.3, we now have a string of algebra isomorphisms...

$$
\begin{array}{c}
H^*(E;R) \\
\uparrow \approx \ \bar{\sigma} \\
\mathrm{Tor}_{R^{\#}(BG)}(R^{\#}(X), R^{\#}(BH)) \\
\uparrow \approx \ \mathrm{Tor}_{\alpha}(1,1) \\
\mathrm{Tor}_{H^*(BG;R)}(R^{\#}(X), R^{\#}(BH)) \\
\downarrow \approx \ T_* \\
\mathrm{Tor}_{H^*(BG;R)}(R^{\#}(X), R^{\#}(BH)) \\
\uparrow \approx \ \mathrm{Tor}_1(1,\beta) \\
\mathrm{Tor}_{H^*(BG;R)}(R^{\#}(X), H^*(BH;R)) \\
\downarrow \approx \ U_* \\
\mathrm{Tor}_{H^*(BG;R)}(R^{\#}(X), H^*(BH;R)) \\
\uparrow \approx \ \mathrm{Tor}_1(\theta_1,1) \\
\mathrm{Tor}_{H^*(BG;R)}(K_1, H^*(BH;R)) \\
\uparrow \approx \ \mathrm{Tor}_1(\theta_2,1) \\
\mathrm{Tor}_{H^*(BG;R)}(K_2, H^*(BH;R)) \\
\uparrow \approx \ \mathrm{Tor}_1(\theta_3,1) \\
\mathrm{Tor}_{H^*(BG;R)}(K_3, H^*(BH;R)) \\
\downarrow \approx \ \mathrm{Tor}_1(\theta_4,1) \\
\mathrm{Tor}_{H^*(BG;R)}(H^*(X;R), H^*(BH;R)) \\
\updownarrow = \\
\mathrm{tor}_{H^*(BG;R)}(H^*(X;R), H^*(BH;R))
\end{array}
$$

This completes the proof of Theorem 4.

THEOREM 5: If

$$\sigma = (E,\pi,X,G/H,G)$$

is a differentiable fibre bundle with X a homogeneous space formed as the quotient G'/H' of a compact, connected Lie group G' by a compact, connected subgroup H' of deficiency 0 in G', then there is an algebra isomorphism

$$H^*(E;R) \approx \mathrm{tor}_{H^*(BG;R)}(H^*(X;R),H^*(BH;R)).$$

PROOF: We recall first the relevant facts about deficiency 0 spaces; Consider

$$H^*(BG';R) = P[x_1,\dots,x_m],$$

$$H^*(BH';R) = P[y_1,\dots,y_n],$$

polynomial algebras on generators of even degree. Consider also the natural map

$$h^*:H^*(BG';R)\to H^*(BH';R)$$

arising from the inclusion of H' into G'. In $H^*(BH';R)$ let I be the ideal generated by $h^*(x_1),\dots,h^*(x_m)$. It may be assumed that the indexing has been chosen so that $h^*(x_1),\dots,h^*(x_p)$ form a non-redundant set of ideal generators for the ideal I. Then the DEFICIENCY of H' in G' is

$$\mathrm{Def}(H',G';R) = p - n.$$

This integer is independent of the choices made in defining it and satisfies

$$0 \leq \mathrm{Def}(H',G',R) \leq \mathrm{Rank}(G') - \mathrm{Rank}(H').$$

If H' has deficiency 0 in G' it follows that the sequence

$$H^*(BG';R) \xrightarrow{h^*} H^*(BH';R) \xrightarrow{k^*} H^*(G'/H')$$

is co-exact. For further details see Baum [1][2].

The proof of Theorem 5 goes through in essentially the same way as the proof of Theorem 4, except that...

The map θ_4 must be redefined. Write K_3 as

$$K_3 = E[u_1,\ldots,u_s] \otimes E \otimes H^*(BH';R),$$

where the elements $u_1,\ldots,u_s$ are not cycles, but E consists of cycles. Define

$$\theta_4(u_i \otimes 1 \otimes 1) = 0,$$

$$\theta_4(1 \otimes 1 \otimes w) = k^*(w),$$

$$\theta_4(1 \otimes w \otimes 1) = [\theta_1\theta_2\theta_3(1 \otimes w \otimes 1)],$$

the class in $H^*(X;R)$ which contains $\theta_1\theta_2\theta_3(1 \otimes w \otimes 1)$ in $R^{\#}(X)$.

The first diagram in our chase is still commutative, this time by our choise of θ_4.

The rest of the argument proceeds as before.

REMARK: From the inequality above it is clear that Theorem 5 is a generalization of Theorem 4. Judging from E. Cartan's list, it appears that Theorem 5 is a generalization of Corollary 2 as well.

REMARK: One might conjecture the existence of an algebra isomorphism

$$H^*(E;R) \approx \text{tor}_{H^*(BG;R)}(H^*(X;R), H^*(BH;R))$$

along the lines of the theorems above in total generality. However, Baum and Smith [3] have given a counterexample.

REMARK: Finally, we remark that all the theorems in this section work with rational coefficients Q as well; we simply use Sullivan's graded commutative rational cochains.

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