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The cohomology of homogeneous spaces in historical context

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ABSTRACT. The real singular cohomology ring of a homogeneous space G/K, interpreted as the Borel equivariant cohomology $H^*_K(G;\mathbb{R})$, was historically the first computation of equivariant cohomology of any nontrivial connected group action. After early approaches using the Cartan model for equivariant cohomology with $\mathbb{R}$ coefficients and the Serre spectral sequence, post-1962 work computing the groups and rings $H^*(G/K;k)$ and $H^*_H(G/K;k)$ with more general coefficient rings motivated the development of minimal models in rational homotopy theory, the Eilenberg–Moore spectral sequence, and A_∞-algebras. In this essay, we survey the history of these ideas and the associated results.

One of the most classical algebraic invariants of a continuous group action $G \curvearrowright X$ is the Borel equivariant cohomology $H^*_G(X;k)$, defined as the singular cohomology of the homotopy orbit space $X_G = (EG \times X)/G$ and studied systematically from 1960. For k the real field $\mathbb{R}$, equivariant cohomology already appears in 1950 in Cartan's work computing the real cohomology ring of a homogeneous space G/K for G a compact, connected Lie group and K a closed, connected subgroup. The determination of the cohomology of a homogeneous space is thus the ur-example of a computation of a ring-valued invariant of a nontrivial connected group action. It was at the same time a motivating example for minimal models in rational homotopy theory.

Generalization of Cartan's result to more general coefficient rings has been similarly fruitful. Such work directly motivated differential homological algebra and the Eilenberg–Moore spectral sequence, and led to substantial development in the field of A_∞- and other up-to-higher-homotopy algebraic structures. This program, which has lasted seventy-five years, is perhaps only now nearing its conclusion. In this article, we will relate the history of these generalizations and the vistas in topology, homological algebra, and homotopy theory that they opened.[1]

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[1] There is much to discuss. This article has been pruned to the satisfy the length restrictions of a proceedings volume and eventually a more lucid update should appear on the arXiv.

In principle, our exposition will assume algebraic topology and homological algebra only up to the basic properties of the Serre spectral sequence (SSS), the spectral sequence of a filtered cochain complex, the universal principal G-bundle $\varsigma = \varsigma_G \colon EG \to BG$, and the ring $\operatorname{Tor}_A(X, Y) = \bigoplus_{p \geqslant 0} \operatorname{Tor}_A^{-p}(X, Y)$ associated to a pair $X \leftarrow A \to Y$ of maps of commutative graded algebras[2]. Comfort with coalgebras and specifically the bar construction will help with later parts of the story but is not strictly necessary for those willing to take some details on faith.

Convention 0.1. We always write G for a compact, connected Lie group, and H and K for closed, connected subgroups. All algebraic objects are cochain complexes (differential d of degree 1[3]) over a commutative base ring k with unity, usually suppressed in the notation $\otimes = \otimes_k$ for the tensor product and $H^*X = H^*(X; k)$ for the singular cohomology ring. Graded modules are cochain complexes with $d = 0$. A ***quasi-isomorphism*** is a cochain map inducing an isomorphism in cohomology. Quasi-isomorphisms induce an equivalence relation on differential graded algebras (allowing zig-zags of maps in alternating directions), and if A' and A lie in the same equivalence class, we say A' is a ***model*** of A. A ***model*** of a space X is a differential graded algebra (DGA) modeling A the cochain algebra $C^*(X) = C^*(X; k)$, so that $H^*(A) \cong H^*(X)$. A smooth manifold X is also modeled by its *de Rham algebra* $A_{\mathrm{dR}}(X)$ for $k = \mathbb{R}$.

The most optimistic guess for a generalization of Cartan's result forms a template which we will adapt as we encounter the actual results:

Theorem 0.2 (one-sided template)**.** *Whenever k is chosen such that $H^*(BG)$ and $H^*(BK)$ are both polynomial rings, there is an isomorphism of graded k-algebras*

$$H^*(G/K; k) \xrightarrow{\sim} \operatorname{Tor}_{H^*(BG)}(k, H^*BK).$$

We call this "one-sided" because it corresponds to the right action of K on G. Given another closed, connected subgroup H, there is also a "two-sided" action of $H \times K$ on G by $(h, x) \cdot g = hgx^{-1}$, leading to a more general guess:

Theorem 0.3 (two-sided template)**.** *Whenever k is chosen such that $H^*(BG)$, $H^*(BH)$, and $H^*(BK)$ are all polynomial rings, there is an isomorphism of graded k-algebras*

$$H^*_H(G/K; k) \xrightarrow{\sim} \operatorname{Tor}_{H^*(BG)}(H^*BH, H^*BK).$$

The actual theorems specialize k in some way, show only an additive isomorphism, come with some restriction on the cochain algebras, or weaken the condition on $H^*(BK)$, but we will see Cartan's progenitor is exactly Theorem 0.2 for $k = \mathbb{R}$.

Most subsequent sections will sketch a proof of a variant of Theorem 0.2 or 0.3. In broad overview, these were first established for k a field of characteristic 0 by Cartan and Borel using the Serre spectral sequence commutative models. In the 1970s they were extended, additively, to more general k, with complications in characteristic 2 using the Eilenberg–Moore spectral sequence and A_∞-algebraic techniques. Multiplicative results for k not containing $\mathbb{Q}$ have come only since 2019, and require 2 to be a unit. The most general multiplicative result is the author's 2021 theorem 22.1 joint with Matthias Franz, while the most general additive result is still Hans J. Munkholm's 1974 theorem 17.12.

2 equipped with a standard grading to be reviewed later

3 N.B.: This is not the case in the primary literature, even in cases motivated primarily by singular cohomology.

Convention 0.4. The degree of a homogeneous element x is written $|x|$. Maps $f\colon C \to A$ of graded k-modules are all k-linear, shifting grading by a fixed degree $|f|$. We write $f \in \mathsf{Mod}_{|f|}(C,A)$. We regard the direct sum $\mathsf{Mod}(C,A) = \bigoplus_{n\in\mathbb{Z}} \mathsf{Mod}_n(C,A)$ as a cochain complex under the differential $Df := d_A f - (-1)^{|f|} f d_C$. A cochain map inducing an isomorphism in cohomology is a ***quasi-isomorphism***.

A DGA is a differential graded k-algebra A equipped with an augmentation $\varepsilon\colon A \longrightarrow k$, with kernel the augmentation ideal $\overline{A} = \ker\varepsilon$. For a cochain algebra $C^*(X)$, an augmentation is induced by restriction to a basepoint in X. Write DGA for the category of DGAs and augmentation-preserving DGA maps. The Koszul sign convention $(f\otimes g)(a\otimes b) = (-1)^{|g||a|} fa\otimes gb$ is always in effect. ***Commutativity*** means graded-commutativity $ab = (-1)^{|a||b|}ba$. A commutative DGA is a CDGA. Commutative graded algebras (CGAs) are CDGAs with $d = 0$. A DGA (or cochain complex) ***computes*** a graded k-algebra (or module) H if its cohomology is H.

A DGA A comes with an augmentation ideal $\overline{A} = \ker\varepsilon$. Its ***ideal of decomposable elements*** is $\overline{AA} = \{\sum a_j b_j : a_j, b_j \in \overline{A}\}$ and its ***module of indecomposables*** is $Q(A) := \overline{A}/\overline{AA}$. Given an oddly graded vector space V of finite type, there is a natural pairing between the exterior algebra $\Lambda[V]$ and the exterior algebra $\Lambda[V^*]$ on the dual $V^* := \mathrm{Hom}_k(V,k)$. The ***primitives*** of $\Lambda[V^*]$ are those elements vanishing on the decomposables $\overline{\Lambda[V]}\cdot\overline{\Lambda[V]}$ of $\Lambda[V]$.

1. Cartan

Cartan's original result from his 1950 announcement [**Cart51**] interprets G_K as the total space of the Borel fibration $G \to G_K \to BK$ and computes $H^*(G/K) = H^*(G_K)$ over $k = \mathbb{R}$.

A key tool is an acyclic CDGA $W(\mathfrak{k})$ equipped with certain operations of the Lie algebra $\mathfrak{k}$, due to Weil (unpublished) and now called the *Weil algebra*, which serves as a universal model for the data associated with a connection on a principal K-bundle. By construction, a connection on a principal K-bundle $\pi\colon E \to B$ corresponds to a unique "characteristic " DGA map $W(K) \longrightarrow A_{\mathrm{dR}}(E)$ preserving the operations of $\mathfrak{k}$. The Weil algebra can also be understood as a model for the total space of the universal principal K-bundle $\varsigma\colon EK \to BK$. Although BK was only defined in full generality by Milnor later (1956), it was understood from known cases that $H^*(BK)$ should be isomorphic to the invariant subring $S[\mathfrak{k}^\vee]^K$ of the symmetric algebra $S[\mathfrak{k}^\vee]$ on the dual $\mathfrak{k}^\vee$ under the coadjoint K-action, graded with $|\mathfrak{k}| = 2$. The CGA underlying the Weil algebra is the tensor product of $S[\mathfrak{k}^\vee]$ and the exterior algebra $\Lambda[\mathfrak{k}^\vee]$, graded with $|\mathfrak{k}^\vee| = 1$, and the inclusion $S(\mathfrak{k}^\vee)^K \longrightarrow W(K)$ can be seen as a model of the universal bundle projection ς. The characteristic map associated to a connection restricts to a map $S[\mathfrak{k}^\vee]^K \to \pi^* A_{\mathrm{dR}}(B) \xrightarrow{\sim} A_{\mathrm{dR}}(B)$, and the induced map in cohomology is the Chern–Weil homomorphism $\chi^*\colon S(\mathfrak{k}^\vee)^K \longrightarrow H^*(B)$.

Given a principal K-bundle $E \to B$, Cartan views $W(K)\otimes A_{\mathrm{dR}}(E)$ as a model for forms on $EK \times E \simeq E$. The projection to $E_K = (EK\times E)/K \simeq E/K = B$ induces a pullback identifying forms on E_K with a subalgebra $\widehat{C}$ of $W(K)\otimes A_{\mathrm{dR}}(E)$ called the *Weil model*. The projection $W(K) = S[\mathfrak{k}^\vee]\otimes\Lambda[\mathfrak{k}^\vee]\otimes A_{\mathrm{dR}}(E) \longrightarrow S[\mathfrak{k}]\otimes A_{\mathrm{dR}}(E)$ induces an algebra isomorphism between $\widehat{C}$ and $C = \big(S[\mathfrak{k}^\vee]\otimes A_{\mathrm{dR}}(E)\big)^K$. Transferring the differential along this isomorphism makes C a DGA called the *Cartan model* computing $H^*(E_K) = H^*_K(E)$.

Results of Hirsch allow one to define subcomplex C' of C, isomorphic to $S[\mathfrak{k}]^K\otimes H^*(E)$, such that the inclusion $C' \hookrightarrow C$ is a quasi-isomorphism. In the special case

$\pi\colon E \to B$ is the quotient projection $G \to G/K$, one can take the ring $A_{\mathrm{dR}}(G)^G \cong \Lambda[\mathfrak{g}^\vee]^G \cong H^*(G)$ as a set of representatives of $H^*(G)$, so that Cartan obtains a DGA structure on $S[\mathfrak{k}^\vee]^K \otimes \Lambda[\mathfrak{g}^\vee]^G$ computing $H^*(G/K)$ as a ring.

Theorem 1.1. *One has the commutative diagram of* CGA*s*

$$\begin{array}{ccccc} & & H^*\big(S[\mathfrak{k}^\vee]^K \otimes \Lambda[\mathfrak{g}^\vee]^G\big) = H^*(C') & & \\ S[\mathfrak{k}^\vee]^K & \hookrightarrow & \downarrow\wr & \to & \Lambda[\mathfrak{g}^\vee]^G = H^*(G). \\ & \xrightarrow{\chi^*} & H^*(G/K) & \xrightarrow{\pi^*} & \end{array}$$

Cartan explores the sequence $S[\mathfrak{g}^\vee]^G \to S[\mathfrak{k}^\vee]^K \to H^*(G/K) \to H^*(G) \to H^*(K)$ and as an example computes the Poincaré polynomials of the real oriented Grassmannians $\widetilde{G}_\ell(\mathbb{R}^{\ell+m}) = \mathrm{SO}(\ell+m)/\big(\mathrm{SO}(\ell)\times\mathrm{SO}(m)\big)$. He also notes that his result implies $H^*(G/K) \cong S[\mathfrak{k}^\vee]^K \otimes_{S[\mathfrak{g}^\vee]^G} \mathbb{R}$ as rings when K is of full rank in G.[4]

Example 1.2. Taking $G = \mathrm{SO}(2n+1)$ and $K = \mathrm{SO}(2)\times\mathrm{SO}(2n-1)$ for $n > 1$ and using restriction relations between Pontrjagin and Euler classes, one finds $H^*\widetilde{G}_2(\mathbb{R}^{2n+1}) \cong \mathbb{R}[e]/(e^{2n})$ for e the image of the universal Euler class in $H^2B\mathrm{SO}(2)$.

The differential of C' in Theorem 1.1 is the unique derivation vanishing on $S[\mathfrak{k}^\vee]^K$ and extending a certain linear map on a space of exterior generators T_G of $\Lambda[\mathfrak{g}^\vee]^G$. Namely, $d|_{T_G} : T_G \to S[\mathfrak{k}^\vee]^K$ is the composite of the restriction $S[\mathfrak{g}^\vee]^G \to S[\mathfrak{k}^\vee]^K$ and a map $\widetilde{\tau}\colon T_G \to S[\mathfrak{g}^\vee]^G$ called a *(choice of) transgression.*

In a cohomological first-quadrant spectral sequence $E_*^{*,*}$, a class of $E_2^{0,p-1}$ is said to ***transgress*** if it survives to $E_p^{0,p-1}$. Then d_p is defined on the class $[z]_p \in E_p^{0,p-1}$, determining an image $\tau[z] = d_p[z]_p \in E_p^{p,0}$ called its ***transgression***. Thus τ is a degree-1 linear map from a submodule of $E_2^{0,*}$ to a quotient of $E_2^{*,0}$. It is often noncanonically lifted to a map $\widetilde{\tau}$ to $E_2^{*,0}$ called a ***choice of transgression***. Cartan's $\widetilde{\tau}$ is that of an algebraic spectral sequence modeled on the SSS of $K \to EK \to BK$, converging from $H^*(BK)\otimes H^*(K) \cong S[\mathfrak{k}^\vee]^K \otimes \Lambda[\mathfrak{k}^\vee]^K$ to $H^*\big(W(K)^K\big) = H^*(EK) = \mathbb{R}$.[5]

Theorem 1.3 (Cartan–Chevalley)**.** *The space T_K of transgressive elements is identical to the space of primitives of $\Lambda[\mathfrak{k}^\vee]^K$, which are exterior generators. The codomain of the transgression is the space $S[\mathfrak{k}^\vee]^K/S^{\geqslant 1}[\mathfrak{k}^\vee]^K S^{\geqslant 1}[\mathfrak{k}^\vee]^K$ of indecomposable elements. Under any choice of transgression $\widetilde{\tau}$ and basis (z_j) for T_K, the images $\widetilde{\tau}z_j$ are irredundant polynomial generators for $S(\mathfrak{k}^\vee)^K$.*[6]

Thus a choice of transgression $\widetilde{\tau}$ induces a linear bijection between exterior generators of $\Lambda[\mathfrak{k}^\vee]^K \cong H^*(K)$ and polynomial generators of $S[\mathfrak{k}^\vee]^K \cong H^*(BK)$.

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[4] This is also a result of Leray which had already been published in the case G is finitely covered by a product of classical groups [**L49**, **L50**].

[5] He actually avoids mentioning spectral sequences as follows. Because the Weil algebra $W(K)$ is acyclic and $\iota\colon S[\mathfrak{k}^\vee]^K \hookrightarrow S[\mathfrak{k}^\vee]\otimes\Lambda[\mathfrak{k}^\vee] \cong W(K)$ is a DGA map, each $x \in S[\mathfrak{k}^\vee]^K$ must be $d_{W(K)}y$ for some $y \in W(K)^K$, which the projection to $\Lambda[\mathfrak{k}^\vee]$ takes to some $z \in \Lambda[\mathfrak{k}^\vee]^K$. The resulting assignment $\sigma\colon S[\mathfrak{k}^\vee]^K \longrightarrow \Lambda[\mathfrak{k}^\vee]^K$ taking x to z is easily seen to be well-defined, and its image is the space T_K of transgressive elements. Cartan's choice of transgression is any linear section $\widetilde{\tau}$ of σ.

[6] Cartan does not include a proof, and notes that this work is inspired in part from Koszul's thesis, which defines Lie algebra cohomology and studies the transgression in a spectral sequence analogous to the SSS of $K \to G \to G/K$, and answers a May 1949 conjecture of Weil.

2. Borel

The proof of Theorem 1.3 relies heavily on the structure of real Lie algebras. Borel's 1952 thesis, among other things, generalizes this transgression theorem as a result about spectral sequences, including over other base fields k.

Theorem 2.1. *Let k be a field and $E_*^{*,*}$ a first-quadrant cohomological spectral sequence of bigraded algebras such that $E_2^{*,*} \cong E_2^{*,0} \otimes E_2^{0,*}$ as a bigraded algebra and $E_\infty^{*,*} = E_\infty^{0,0} = k$.*

- *If $E_2^{0,*}$ is an exterior algebra on odd-degree elements, then there exist homogeneous transgressive z_j such that $E_2^{0,*} = \Lambda[z_j]$.*
- *If $\operatorname{char} k = 2$, suppose there exist homogeneous transgressive $z_1, \ldots, z_n \in E_2^{0,*}$ (of any degree) such that the monomials $z_{j_1} \cdots z_{j_\ell}$ ($j_1 < \cdots < j_\ell$, $\ell \leqslant n$) form a basis of $E_2^{0,*}$.*

In either case, for any choice $\widetilde{\tau}$ of transgression, $E_2^{*,0} = k[\widetilde{\tau} z_j]$.[7]

In the rest of this section we consider a bundle $F \xrightarrow{i} E \xrightarrow{\varpi} B$ with trivial $\pi_1(B)$-action on $H^*(F)$. In the SSS of this bundle, the transgression goes from $H^*(F)$ to $H^*(B)$.[8] Applied to $K \to EK \xrightarrow{\varsigma} BK$, Theorem 2.1 says that if $H^*(K)$ is exterior over a field k of characteristic $\neq 2$, then $H^*(BK)$ is polynomial on a basis of transgressions.[9] In retrospect, this proves Theorem 0.2 for $K = 1$ and $G/K = G$; in dealing with $H^*(G)$, Borel is also the first to characterize finitely-generated commutative Hopf algebras over $\mathbb{F}_p$. He also shows that in the SSS of ς, if k is such that $H^*(K)$ is a free module on monomials in transgressive generators as in Theorem 2.1, the map $H^*(K) \to H^*(K \times K) \xrightarrow{\sim} H^*(K) \otimes H^*(K)$ induced by the group multiplication $K \times K \to K$ takes z to $1 \otimes z + z \otimes 1$ precisely for these transgressive generators. These elements are also called ***primitive***, and this agrees with the previous notion if $H^*(K)$ is exterior.

To compute $H^*(G/K)$, we want a functorial $\mathbb{R}$-CDGA $A(-)$ computing cohomology of spaces. The de Rham algebra $A_{\mathrm{dR}}(X)$ does this only for manifold X.[10] Sullivan later introduced the $\mathbb{Q}$-algebra A_{PL} of polynomial differential forms, and $k \otimes_{\mathbb{Q}} A_{\mathrm{PL}}(-)$ works for all fields k of characteristic 0. Equipped with such a model, Borel can use the SSS to generalize Cartan's model C' from Theorem 0.2. **From now on, suppose additionally $H^*(F)$ is exterior on a set of generators transgressing in the SSS.** If $F = G$ and ϖ is a principal G-bundle, Theorem 2.1 shows this happens if $\operatorname{char} k \neq 2$ or $\operatorname{char} k = 2$ and the generators z_j transgress in the SSS of the universal bundle: indeed, the classifying map χ induces a map of SSSs from that of ς to that of ϖ, so that in the latter, each z_j transgresses to $\tau_\varpi(z_j) = \chi^* \tau_\varsigma(z_j) \in H^*(B)$.

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[7] This can be strengthened by requiring only $\bigoplus_{p+q=1}^{n} E_\infty^{p,q} = 0$ and concluding only $E_2^{\leqslant n,0}$ is polynomial, where $\max_j |z_j| \leqslant \frac{n}{2} - 1$.

[8] Unpacking the definition, if a cocycle $z \in C^{p-1}(F)$ represents a transgressive class $[z] \in H^{p-1}(F) = E_2^{0,p-1}$, then there exist a cochain $y \in C^{p-1}(E)$ with $i^* y = z$ and a cocycle $x \in C^p(B) = E_2^{p,0}$ such that $\pi^* x = \delta y$, and then $d_p[z]_p = [x]_p$.

[9] The degree-truncated version of Borel's transgression theorem is relevant because Borel and Cartan used only finite-dimensional truncations of BG, which would later be defined for general topological groups G by Milnor.

[10] Borel extends this to compact separable metric spaces X of finite dimension, using the Menger–Nöbeling theorem asserting a homeomorphic embedding $X \rightarrowtail \mathbb{R}^{1+2\dim X}$, restricting the sheaf $U \mapsto A_{\mathrm{dR}}(U)$ on $\mathbb{R}^N$ to X and taking global sections.

Borel endows the graded algebra $L = A(B) \otimes H^*(F)$ with the unique derivation extending that on $A(B) \cong A(B) \otimes \mathbb{R}$ and taking each $1 \otimes z_j$ to a cocycle in $A(B) \otimes \mathbb{R}$ representing $\tau z_j \otimes 1 \in H^*(B) \otimes \mathbb{R}$. For each z_j, one can find an $A(E)$-cochain y_j restricting along i to representative of z_j in $A(F)$, and such a choice induces an algebra map $j\colon H^*(F) \to A(E)$ by the commutativity of $A(E)$. There is thus a DGA map $L \longrightarrow A(E)$ taking $b \otimes y \longmapsto \varpi^*(b) \cdot j(y)$.

Theorem 2.2 ([**Bo53**, Thm. 24.1′])**.** *Under the* DGA *structure on* $L = A(B) \otimes H^*(F)$ *induced by the transgression as in the previous paragraph, the map* $L \to A(E)$ *induces a* k-CGA *isomorphism* $H^*\big(A(B) \otimes H^*(F)\big) \xrightarrow{\sim} H^*(E)$.

Proof. Replacing B by a weakly equivalent CW complex if necessary, $A(E)$ inherits the Serre filtration $F_pA(E) = \ker\big(A(E) \to A(\varpi^* B_{p-1})\big)$ and $A(B) \otimes H^*(F)$ the filtration by $\ker\big(A(B) \to A(B_{p-1})\big) \otimes H^*(F)$.[11] The map $L \to A(E)$ respects these filtrations, inducing a map of spectral sequences which reduces on E_2 pages to $\mathrm{id}_{H^*(B) \otimes H^*(F)}$ and hence is a quasi-isomorphism. □

Suppose additionally from now on that there exists a sub-DGA H of $A(B)$ such that the inclusion is a quasi-isomorphism. Then one can select representatives of $\tau z \otimes 1 \in H^*(B) \otimes \mathbb{R}$ lying in $H \otimes \mathbb{R}$ to define a sub-DGA $L' = H \otimes H^*(F)$ of L, and another filtration spectral sequence argument shows $L' \hookrightarrow L$ is a quasi-isomorphism, so the composite $L' \to A(E)$ is as well.

Theorem 2.3. *n/*[**Bo53**, Thm. 25.1]*/*[12] *Under the* DGA *structure on* L' *induced by the transgression as in the previous paragraphs, the map* $L \to A(E)$ *induces a* k-CGA *isomorphism* $H^*\big(H^*(B) \otimes H^*(F)\big) \xrightarrow{\sim} H^*(E)$.

The hypotheses of Theorem 2.3 are always satisfied for $E = F_K \longrightarrow BK = B$ the Borel fibration associated to the action of a compact, connected Lie group K-action on F: any choice of A-cocycles representing generators of the polynomial ring $H^*(BK)$ induces a unique DGA map $H^*(BK) \longrightarrow A(BK)$, and we may take for H its image, yielding an isomorphism $H^*\big(H^*(BK) \otimes H^*(F)\big) \longrightarrow H^*(F_K)$. Taking $F = G$ a compact Lie group containing K, with the translation action, we get back Cartan's isomorphism $H^*\big(H^*(BK) \otimes H^*(G)\big) \longrightarrow H^*(G/K)$ from Theorem 0.2 [**Bo53**, Thm. 25.2].

Borel is able to obtain results when $\operatorname{rk} K = \operatorname{rk} G$ extending Leray's results over a field of characteristic p or $\mathbb{Z}$: if T is a maximal torus of K and the cohomology of G, K, G/T, and K/T are assumed to be p-torsion–free for characteristic p or torsion-free for $k = \mathbb{Z}$, then $H^*(G/K) \cong H^*(BK) \otimes_{H^*(BG)} k$. He provides several explicit computations, applying the equal-rank result to the cohomology rings of $\mathrm{U}(\sum n_j)/\prod \mathrm{U}(n_j)$ and $\mathrm{Sp}(\sum n_j)/\prod \mathrm{Sp}(n_j)$ for $k = \mathbb{Z}$ and to $\mathrm{SO}(2n)/\mathrm{U}(n)$ for $\operatorname{char} k \neq 2$. These computations use Chevalley's restriction isomorphisms $H^*(BG) \longrightarrow H^*(BT)^{W_G}$ for G and K, for W_G the Weyl group, which holds assuming the torsion conditions are satisfied. For $\operatorname{char} k = 2$, he shows $\mathrm{SO}(2n) \twoheadrightarrow \mathrm{SO}(2n)/\mathrm{U}(n)$ induces an injection in cohomology onto the subalgebra generated by even-degree primitives.

Other spaces treated are $\mathrm{U}(2n)/\mathrm{Sp}(n)$ for $k = \mathbb{Z}$ and $\mathrm{U}(n)/\mathrm{SO}(n)$ for $\operatorname{char} k \neq 2$. In later work he deals with the same space and with $\mathrm{O}(\sum n_j)/\prod \mathrm{O}(n_j)$ for

[11] This is a simplification; Borel's actual filtration is essentially by degree of forms.

[12] Borel restricts to principal G-bundles in his statement, using his Thm. 24.1, but using Thm. 24.1′ instead (Theorem 2.2), the restriction is unnecessary.

$\operatorname{char} k = 2$, using instead of a torus T the diagonal elementary abelian 2-group. Using the already-studied SSS of $K \to G \to G/K$ instead, he also analyzes the Stiefel manifolds $\mathrm{U}(n)/\mathrm{U}(\ell)$ and $\mathrm{Sp}(n)/\mathrm{Sp}(\ell)$ for $\ell < n$ and $k = \mathbb{Z}$, and $\mathrm{SO}(n)/\mathrm{SO}(\ell)$ in characteristic 2 and additively for $k = \mathbb{Z}$.

Example 2.4. Taking $G = \mathrm{U}(n)$ and $K = \mathrm{U}(1)^n$ diagonal, one finds $H^*(G/K;\mathbb{Z}) \cong \mathbb{Z}[t_1, \ldots, t_n]/\big(\sum t_j, \sum_{i<j} t_i t_j, \sum_{h<i<j} t_h t_i t_j, \cdots, t_1 \cdots t_n\big)$, where $|t_j| = 2$.

3. Eschenburg

There has long been a school of geometers interested in compact Riemannian manifolds of positive curvature. There seem to be relatively few of these, and many known examples are ***biquotients***, orbit spaces of a Lie group G under a free "two-sided" action $(u, v)\cdot g = ugv^{-1}$ of a subgroup U of $G \times G$. This obviously specializes to a homogeneous space for $U = 1 \times K$. The special case where $U = H \times K$ for two closed subgroups $H, K \leqslant G$ is written $H\backslash G/K$, and this case is in fact general, since writing $\Delta\colon G \to G \times G$ for the diagonal, $(x, y) \mapsto xy^{-1}\colon G \times G \to G$ induces a natural diffeomorphism $G/U = U\backslash(G \times G)/\Delta G$.

Eschenburg [**Esc92**] studied the cohomology of biquotients, showing many aspects of Borel's and Cartan's analyses generalize. For convenience, given a right G-space X and left G-space Y, write $X \otimes_G Y$ for the orbit space under $(xg, y) \sim (x, gy)$. Consider the two-sided action of $G \times G$ on G and the restricted action of U. Eschenburg notes the following system of quotients of $EG \times EG \times G$:[13]

$$\begin{array}{ccc} G_U \simeq (EG \times EG) \otimes_U G & \longrightarrow & (EG \times EG) \otimes_{G\times G} G \approx (EG \times EG)/\Delta G \\ \Big\downarrow {\scriptstyle\varpi} & & \Big\downarrow {\scriptstyle\delta} \\ BU \simeq (EG \times EG) \otimes_U * & \xrightarrow[\chi]{} & (EG \times EG) \otimes_{G\times G} * \approx BG \times BG. \end{array} \tag{3.1}$$

Here the upper-left corner is of interest because it is homotopy equivalent to U when the action is free. Since the fiber of both ϖ and δ is G, this is a bundle map, inducing a map of SSSs. Because δ can be identified with the diagonal map $BG \longrightarrow BG \times BG$ up to homotopy, $H^*(\delta)$ can be identified with the cup product on the polynomial algebra $H^*(BG)$, a surjection with kernel generated by the elements $1 \otimes \tau z_j - \tau z_j \otimes 1 \in H^*(BG) \otimes H^*(BG) \cong H^*(BG \times BG)$ for z_j generators of the exterior algebra $H^*(G)$ and $\tau = \tau_\varsigma$ a choice of transgression for $G \to EG \xrightarrow{\varsigma} BG$. Since none of the z_j survive the SSS of δ, it follows each transgresses to $\tau_\delta(z_j) = 1 \otimes \tau z_j - \tau z_j \otimes 1$. Applying the map to the SSS of ϖ, one finds the following:

Proposition 3.1 (Eschenburg)**.** *Each z_j transgresses to $\tau_\varpi(z_j) = \chi^* \tau_\delta(z_j)$.*

Eschenberg uses this to study the SSS of ϖ and compute several examples.

4. Kapovitch

Eschenburg noted his G-bundle map $\varpi \to \delta$ implies exterior generators of $H^*(G)$ transgress in the SSS of ϖ. Taking $U = H \times K$, in his thesis work in 2014,

[13] The only non-obvious equivalence may be the homeomorphism $(e, e') \otimes x \mapsto (ex, e')\Delta G$ on the upper right. It is well-defined because $(eg, e'g') \otimes g^{-1}xg' = (e, e') \otimes x$ is also sent to $(eg \cdot g^{-1}xg', e'g')\Delta G = (exg', e'g')\Delta G = (ex, e')\Delta G$. It is obviously surjective. It is injective because if $(fy, f')\Delta G = (ex, e')\Delta G$, there is $g \in G$ with $(fyg, f'g) = (ex, e')$, so $e = fygx^{-1}$, and $(e, e') \otimes x = (fygx^{-1}, f'g) \otimes x = (fyg, f'g) \otimes x^{-1}x1 = (fy, f') \otimes g^{1-}1g = (f, f') \otimes y$.

the author noticed that Theorem 2.3, which is essentially in Borel's thesis, yields a model $H^*(BH \times BK) \otimes H^*(G)$ for $H^*(G_{H\times K}) \cong H^*_H(G/K)$. It turns out this had been known for ten years for more general reasons.

Borel's quasi-isomorphism $L' \longrightarrow A(G_K)$ can be seen as an early example of rational homotopy theory in action (and so, with more squinting, the results of Cartan, Chevalley, Koszul, and Weil can be as well). A *Sullivan algebra* is CDGA over a field k of characteristic 0 which is free (exterior $\otimes$ polynomial) as a CGA and whose differential satisfies a certain nilpotence property, and one constructs and computes with *Sullivan models* of spaces, Sullivan algebras computing their cohomology. Borel's theorem 2.3 applies to the universal G-bundle $\varsigma\colon EG \to BG$ by his transgression theorem 2.1, defining a model $H^*(BG) \otimes H^*(G)$ for EG with differential the derivation defined as 0 on $H^*(BG)$ and on exterior generators by $z_j \longmapsto \tau z_j$. This is a Sullivan model for EG. The bundle $G_K \to BK$ is the pullback of ς under the classifying map $\rho = B(K \hookrightarrow G)$, and $H^*(\rho)\colon H^*(BG) \to H^*(BK)$ is a map of Sullivan models, modeling ρ, and the inclusion $H^*(BG) \hookrightarrow H^*(BG) \otimes H^*(G)$ models ς. A standard result on Sullivan models [FHT01, §15(c)] says in essence that given a map $X \to B$ to a simply-connected B, a Serre fibration $E \to B$, and Sullivan models $M_X \leftarrow M_B \hookrightarrow M_E$, finitely generated in each degree, and such that $M_B \hookrightarrow M_E$ is a *relative Sullivan algebra*, meaning the map of underlying CGAs is $M_B \hookrightarrow M_B \otimes A$ for some other CGA A, the tensor product $M_X \otimes_{M_B} M_E$ is a Sullivan model for the pullback $X \times_B E$. Applying this, we get back Borel's model for G_K:

$$H^*(BK) \underset{H^*(BG)}{\otimes} \big(H^*(BG) \otimes H^*(G)\big) \cong H^*(BK) \otimes H^*(G) = L'. \tag{4.1}$$

Building on Eschenburg's work, Kapovitch notes $H^*(\chi)\colon H^*(BH) \otimes H^*(BK) \to H^*(BG) \otimes H^*(BG)$ is a Sullivan model of $\chi = B(H \times K \hookrightarrow G \times G)$ and Eschenburg's transgression result yields a Sullivan model $R = H^*(BG) \otimes H^*(G) \otimes H^*(BG)$ of $(EG \times EG)/\Delta G$ with differential vanishing on $H^*(BG) \otimes H^*(BG)$ and defined by the transgressions $\tau_\delta(z_j) \longmapsto 1 \otimes \tau z_j - \tau z_j \otimes 1$ on exterior generators z_j of $H^*(G)$. Moreover the inclusion $H^*(BG) \otimes H^*(BG) \hookrightarrow R$ is a model of δ. Then the same result on pullbacks and Sullivan models gives the following.

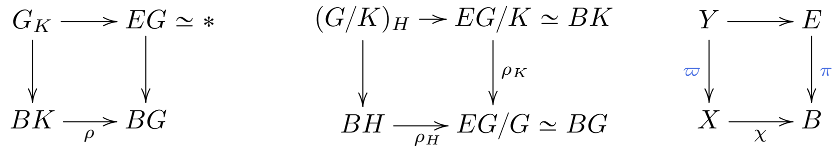
Theorem 4.1 ([Kap04])**.** *The equivariant cohomology $H^*_H(G/K)$ over $k = \mathbb{Q}$ is the cohomology of the Sullivan algebra*

$$\big(H^*(BH) \otimes H^*(BK)\big) \underset{H^*(BG) \otimes H^*(BG)}{\otimes} R \cong H^*(BH) \otimes H^*(BK) \otimes H^*(G)$$

with differential the derivation vanishing on $H^*(BH) \otimes H^*(BK)$ and defined on exterior generators of $H^*(G)$ by $z_j \longmapsto 1 \otimes \rho_K^* \tau z_j - 1 \otimes \rho_H^* \tau z_j$ for $\rho_H = B(H \hookrightarrow G)$ and $\rho_K = B(K \hookrightarrow G)$.

Example 4.2 ([Ca21, Prop. 1.9])**.** Let $G = \mathrm{SU}(n)$ and $K \cong \mathrm{U}(1)$ a *reflected circle*, meaning that for some $g \in G$, for all $x \in K$, one has $gxg^{-1} = x^{-1}$. Then $H^*_K(G/K) \cong \Lambda[z_5, \ldots, z_{2n-1}] \otimes \mathbb{Q}[s,t]/(s^2 - t^2)$, where $|s| = |t| = 2$ and $H^*(G) \cong \Lambda[z_3, z_5, \ldots, z_{2n-1}]$.

Example 4.3 ([He16][Ca21])**.** For $G = \mathrm{SO}(\ell + m)$ and $K = \mathrm{SO}(\ell) \times \mathrm{SO}(m)$, so that G/K is the oriented Grassmannian, the ring $H^*_K(G/K; \mathbb{Q})$ has been expressed in work of the author using the Kapovitch model, following a computation by different means by Chen He.



(A) (B) (C)

FIGURE 5.1

5. The Eilenberg–Moore spectral sequence

A critical feature of the approaches described so far is the use of abstractions of differential forms to give CDGA models of spaces over fields of characteristic 0. Functorial CDGA models do not exist in other characteristics,[14] but one aspect of the computation does generalize.

The model $H^*(BG)\otimes H^*(G)$ of EG from the previous section, regraded by exterior degree, is a free $H^*(BG)$-module resolution of $k \cong H^*(BG)/H^{\geqslant 1}(BG)$, so Borel's model $H^*(BK)\otimes H^*(G)$ for G/K computes $\mathrm{Tor}_{H^*BG}(k, H^*BK)$. This finally explains why we have stated Cartan's theorem 1.1 as Theorem 0.2 for $k = \mathbb{R}$. The spaces of interest fit into (5.1a).

We obtain Theorem 0.3 for $k = \mathbb{Q}$ from Theorem 4.1 in similar fashion. The Sullivan model R of BG from the previous section, viewed as a left $H^*(BG)$-module, yields an (inefficient) $H^*(BG)$-module resolution of $H^*(BG)$ itself via the cup product $R_0 = H^*(BG)\otimes H^0(G)\otimes H^*(BG) \longrightarrow H^*(BG)$. Thus $R\otimes_{H^*(BG)} H^*(BK)$ computes $\mathrm{Tor}_{H^*(BG)}(H^*BG, H^*BK) = H^*(BK)$, and so $R\otimes_{H^*(BG)} H^*(BK)$ is an $H^*(BG)$-module resolution of $H^*(BK)$. Thus $\mathrm{Tor}_{H^*(BG)}(H^*BH, H^*BK)$ can be computed as the cohomology of

$$H^*(BH) \underset{H^*(BG)}{\otimes} R \underset{H^*(BG)}{\otimes} H^*(BK) \cong H^*(BH)\otimes H(G)\otimes H^*(BK),$$

which is the Kapovitch model of $G_{K\times H}$ from Theorem 4.1.[15] This can also be recovered through (5.1b), which is pullback square because of the homeomorphism $EG\otimes_H G/K \longrightarrow EG/H \times_{EG/G} EG/K$ given by $e\otimes gK \longmapsto (eH, egK)$. If we take $\rho_H^* \colon H^*(BG) \to H^*(BH)$ as a Sullivan model of ρ_H and the left module structure

[14] The proof from Borel's 1951 ETH lectures on the Leray spectral sequence is as follows [**Bo51**, Thm. 7.1]. Suppose for a contradiction that A is a k-CDGA–valued contravariant functor on topological spaces, for k a ring of characteristic $p > 0$, such that $H^*\big(A(-)\big) \cong H^*(-)$ and $i^* \colon A(Y) \longrightarrow A(X)$ is surjective whenever $i \colon X \longhookrightarrow Y$ is the inclusion of a closed subset.

We show this impossible for $X = \mathbb{C}\mathrm{P}^n$ with $n > p$ and $Y \simeq *$ the cone on X. Note that for any even-degree y in a k-CDGA, one has $d(y^p) = py^{p-1}dy = 0$. Let the cocycle $a \in A^2(X)$ represent a generator x of $H^*(X) \cong k[x]/(x^{n+1})$, which since X is closed in Y is $i^*\tilde{a}$ for some $\tilde{a} \in A^2(Y)$. Now $\tilde{a}^p \in A^{2p}(Y)$ is a cocycle, and since Y is contractible, also a coboundary. But then as i^* is a DGA map, $i^*(\tilde{a}^p) = a^p$ is also a coboundary despite representing $x^p \neq 0$ in $H^{2p}(X)$.

[15] We promised we would discuss the grading. A projective resolution $P_\bullet \longrightarrow M$ of an A-module M is a sequence of degree-0 A-module maps $P_p \longrightarrow P_{p+1}$ for $p \leqslant 0$ together with a degree-0 A-module map $P_0 \longrightarrow M$ such that the sequence $\cdots \to P_{-1} \to P_0 \to M \to 0$ is exact. A pure tensor $x\otimes y$ of homogeneous elements $x \in P_p$ and $y \in N$ inherits a well-defined "internal" degree $q = |x| + |y|$ in addition to the nonpositive resolution degree p, inducing a bigrading on $P_\bullet \otimes_A M$ and hence on the cohomology $\mathrm{Tor}_A(M, N)$ of the resulting single complex. It is the total degree $n = p + q$ that is the relevant grading for Theorem 0.3 and Theorem 0.2.

map $H^*(BG) \hookrightarrow R \otimes_{H^*(BG)} H^*(BK)$ as a Sullivan model of ρ_K, then the standard result on pullbacks again gives Theorem 4.1.

Both (5.1a) and (5.1b) are pullbacks of fibrations (5.1c). The common features of our hypotheses are that k is a field of characteristic 0 that $H^*(\chi)$ itself is a Sullivan model for χ, and that the Sullivan model M_E for E is an $H^*(X)$-module resolution of $H^*(E)$ in such a way that $\pi\colon H^*(X) \to M_E$ is the module structure map. The common conclusion is a ring isomorphism $\mathrm{Tor}_{H^*(B)}(H^*X, H^*E) \to H^*(Y)$.

We would like something like this to hold more generally, but in general, one cannot hope to find models with trivial differential, or in other characteristics, even spaces with polyonomial cohomology. In general, one at least has a map $\mathrm{Tor}^0_{H^*(B)}(H^*X, H^*E) = H^*(X) \otimes_{H^*(B)} H^*(E) \longrightarrow H^*(Y)$, which is an isomorphism if B is contractible and k is a field, by the Künneth theorem. We still have cochain algebras and a map $C^*(X) \otimes_{C^*(B)} C^*(E) \longrightarrow C^*(Y)$ for all k, and this is again a quasi-isomorphism when B is contractible, by the Eilenberg–Zilber theorem, but usually not otherwise. To generalize the description of the cohomology of a pullback to cases with noncommutativity, nonzero differentials, and nonzero B, Eilenberg and Moore consider a a refined notion of resolution.

Definition 5.1. Given a DGA A, a ***DG A-module*** M is a DG k-module M which is simultaneously an A-module in such a way that the action map $A \otimes M \longrightarrow M$ is a cochain map (meaning $d(am) = da \cdot m + a \cdot (-1)^{|a|} dm$). A map of DG A-modules is an A-module map which is also a degree-0 cochain map. A sequence of DG A-module maps $P_p \to P_{p+1}$ is ***proper exact*** if for each fixed degree j the three sequences $P^j_\bullet$, $B^j(P_\bullet)$, and $H^j(P_\bullet)$ of k-modules are exact. A DG A-module P is ***proper projective*** if for each proper exact sequence $M \xrightarrow{f} N \to 0$ of A-modules, each DG A-module map $g\colon P \longrightarrow N$ lifts along f.[16] Given a DG A-module M, there always exists a ***proper projective resolution*** of M, a sequence of proper projective DG A-modules $P_\bullet = (P_p)_{p \leqslant 0}$ and a DG A-module map $P_0 \to M$ such that the extended sequence $P_\bullet \to M \to 0$ is proper exact. This guarantees $H^*(P_\bullet)$ be a projective $H^*(A)$-module resolution of $H^*(M)$.[17]

If M and N are DG A-modules and $P_\bullet$ a proper projective resolution of M, then as in the classical case, $P_\bullet \otimes_A N$ inherits an internal grading q defined on pure tensors by $q = |x \otimes y| = |x| + |y|$, the resolution grading p, and the total grading $n = p + q$. The ***differential Tor*** $\mathrm{Tor}_A(M, N)$, defined as the cohomology of the associated single complex, with the inherited bigrading, does not depend on the choice of $P_\bullet$.

Returning to Figure 5.1c, a proper projective resolution $P_\bullet$ of the $C^*(B)$-module $C^*(X)$ comes with a surjection $P_\bullet \twoheadrightarrow P_0 \twoheadrightarrow C^*(X)$ inducing a composite

$$P_\bullet \underset{C^*(B)}{\otimes} C^*(E) \longrightarrow C^*(X) \underset{C^*(B)}{\otimes} C^*(E) \longrightarrow C^*(Y). \tag{5.1}$$

---

[16] *I.e.*, there exists a DG A-module map $\tilde{g}\colon P \longrightarrow M$ with $f\tilde{g} = g$.

[17] For $A = k$, this recovers the notion of a *Cartan–Eilenberg resolution* of a complex [CE56, Ch. XVII], used to define hypercohomology: given an additive functor F, one takes the single complex P associated to $P_\bullet$ and defines the hypercohomology of M with respect to F to be $H^*F(P)$. To relate this to later terminology, in one of the model structures on the category of half-plane bicomplexes $C^{\bullet,\bullet}$ a cofibrant replacement of a single complex $C^{0,\bullet}$ is exactly a Cartan–Eilenberg resolution [MuR19, §4], so that hypercohomology with respect to F is the derived functor of F.

Theorem 5.2 (Eilenberg–Moore)**.** *Suppose that k is a principal ideal domain, $\pi_1(B)$ acts trivially on the homotopy fiber F of $E \longrightarrow B$, and either (a) each of the k-modules $H^n(B)$ and $H^n(X)$ is finitely generated or (b) each $H^n(F)$ is. Then* (5.1) *induces an isomorphism*

$$\mathrm{Tor}_{C^*(B)}(C^*X, C^*E) \xrightarrow{\sim} H^*(Y),$$

which is multiplicative under a certain natural ring structure on the domain.

Sketch. Replace everything with a CW-complex and $X \longrightarrow B$ with a cellular map, Filter $C^*(Y)$ of (5.1) by the X-skeletal Serre filtration $F_pC^*(Y) = \ker\big(C^*(Y) \to C^*(\varpi^* X_{p-1})\big)$ and $C^*(E)$ by the B-skeletal Serre filtration, $P^\bullet$ by resolution degree, and the domain of (5.1) by the tensor filtration. Then (5.1) is filtration-preserving and hence induces a map of filtration spectral sequences, whose codomain is the SSS converging to $H^*(Y)$ and whose E_2 page map unwinds as the identity map of $H^*(X; H^*F)$. Thus (5.1) is a quasi-isomorphism. □

If we instead filter $P_\bullet \otimes_{C^*(B)} C^*(E)$ or more generally $P_\bullet \otimes_A N$ by the resolution degree p, we get a left–half-plane spectral sequence of Künneth type, the ***algebraic Eilenberg–Moore spectral sequence***, with $E_2 = \mathrm{Tor}_{H^*(A)}(H^*M, H^*N)$, converging to $\mathrm{Tor}_A(M, N)$.

Corollary 5.3 (***Eilenberg–Moore spectral sequence*** (EMSS) [**EMo65**, **Sm67**])**.** *Under the hypotheses of Theorem 5.2, there exists a spectral sequence of $H^*(B)$-algebras converging to $H^*(Y)$ with $E_2 = \mathrm{Tor}_{H^*(B)}(H^*X, H^*E)$.*[18]

There is typically no easy way to compute the differentials of the EMSS unless it is known to collapse, so many authors have set themselves the task of proving EMSS collapse results. A recurrent strategy runs through the algebraic EMSS:

Proposition 5.4 ([**Mac**, XI.3.2][**GM**, Cor. 1.8][**Mun74**, Theorem 5.4])**.** *The algebraic* EMSS *associated to a diagram $M \longleftarrow A \longrightarrow N$ of maps of nonnegatively-graded* DGA*s is convergent and functorial in the sense that a commutative diagram*

$$\begin{array}{ccccc} M' & \xleftarrow{\phi_{M'}} & A' & \xrightarrow{\phi_{N'}} & N' \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle v} \\ M & \xleftarrow[\phi_M]{} & A & \xrightarrow[\phi_N]{} & N \end{array} \tag{5.2}$$

of DGA *maps induces a map of spectral sequences. In particular, if f, u, v are quasi-isomorphisms, the induced map $\mathrm{Tor}_f(u, v)$ is a graded-linear isomorphism.*

[18] The earliest version of differential homological algebra and hence of the Eilenberg–Moore spectral sequence was worked out by Eilenberg and Moore as early as 1957; per a 1959 lecture of Moore in the *Séminaire Henri Cartan* [**Mo60**, fn. 1], it was a topic of the Princeton topology seminar in 1957–8. The cohomological version of differential homological algebra and the EMSS appears in full in the unpublished 1962 version of Paul Baum's thesis [**Baum**]. The first account of the cohomological EMSS published in a journal seems to have been Larry Smith's from 1967 [**Sm67**]. The homological version, with Cotor and coalgebras in place of Tor and algebras, appears in the Eilenberg–Moore paper only in 1965 [**EMo65**], the cohomological version being deferred to a Part II yet to appear. Reference to the related bar spectral sequence appears in work of Clark [**Cl65**] to be discussed later.

Thus if we can find vertical maps making the diagram

$$
\begin{array}{ccccc}
H^*(X) & \leftarrow & H^*(B) & \rightarrow & H^*(E) \\
\downarrow & & \downarrow & & \downarrow \\
C^*(X) & \leftarrow & C^*(B) & \rightarrow & C^*(E),
\end{array}
\tag{5.3}
$$

commute, we can conclude $H^*(Y) \cong \mathrm{Tor}_{C^*(B)}(C^*X, C^*E) \cong \mathrm{Tor}_{H^*(B)}(H^*X, H^*E)$, a strong EMSS collapse result, and in the case that Figure 5.1c is Figure 5.1a or Figure 5.1b, we obtain Theorem 0.2 or Theorem 0.3, respectively. We usually cannot show (5.3) is commutative on the nose, but we will encounter more general versions of Tor, each admitting a functorial algebraic EMSS and an analogue of Proposition 5.4, and for these generalizations, we will be able to follow this strategy.

6. Baum

Paul Baum's 1962 thesis [**Baum**] aimed to establish Theorem 0.2 for k a field by proving the collapse of the EMSS of Figure 5.1a. For T a maximal torus of K, he noted there is a map of fibrations from $G/T \to BT \to BG$ to $G/K \to BK \to BG$, inducing a map of EMSSs which he showed was of the form $E_r \hookrightarrow E_r \otimes H^*(K/T)$. As a consequence, one has the following.

Theorem 6.1 (Baum [**Baum**, 3.3.2])**.** *Let k be a field. If T is a maximal torus of K, the* EMSS *converging to $H^*(G/K)$ collapses if and only if that converging to $H^*(G/T)$ does.*

Later proofs often call on mild variants of Theorem 6.1; for the purposes of this survey, we will gloss all of them as ***Baum's reduction***.

Unfortunately, there is an error in the proof of the main collapse result.[19] In the 1968 published version [**Baum68**], the following is salvaged.

Theorem 6.2 (Baum)**.** *For any field k, the* EMSS *associated to Figure 5.1a collapses when the kernel of the map $QH^*(BG) \to QH^*(BT)$ of indecomposables has dimension $\leqslant 2$. In particular, Theorem 0.2 holds additively.*

PROOF. The EMSS is concentrated in even rows q, forcing $d_2 = 0$. By the assumption on indecomposables, it is generated in columns $-2 \leqslant p \leqslant 0$, forcing $d_{\geqslant 3} = 0$. □

In particular, when G and K are of equal rank, $H^*(BG) \longrightarrow H^*(BK)$ is surjective so the EMSS is concentrated in the 0^{th} column, forcing a CGA isomorphism $H^*(G/K) \xrightarrow{\sim} k \otimes_{H^*(BG)} H^*(BK)$ recovering Borel's result [**Baum68**, Cor. 7.5].

Example 6.3 (Borel)**.** Consider $K = \mathrm{SU}(5) < \mathrm{U}(5) < \mathrm{Sp}(5) = G$. Computing the map $H^*B\mathrm{Sp}(5) \longrightarrow H^*B\mathrm{SU}(5)$ and computing Tor using a Koszul complex, one has $H^*(G/K) = \mathbb{F}_p\{1, c_3, c_5, w_{21}, w_{26}, [G/K]\}$, where $c_j \in H^{2j}(G/K)$ are the images of the universal Chern classes, $\dim G/K = 31$, and the only nonzero products are those implied by Poincaré duality, for $k = \mathbb{F}_p$ $(p \neq 2)$ or $k = \mathbb{Q}$. This looks different than the previous results because the ideal of $H^*B\mathrm{SU}(5)$ generated by $H^{\geqslant 1}B\mathrm{Sp}(5)$ is not generated by a regular sequence, and equivalently (for $k = \mathbb{Q}$),

[19] Paul will readily tell you this if you happen to coincidentally sit next to him at a conference and tell him you've just read his thesis.

G/K is not formal in the sense of rational homotopy theory. For $k = \mathbb{F}_2$, one instead finds $H^*(G/K) \cong \Lambda[z_3] \otimes \mathbb{F}_2[c_2, c_3, c_4, c_5]/(c_2^2, c_3^2, c_4^2, c_5^2)$.

Counterexample 6.4. The center of the unitary group $\mathrm{U}(2)$ is the diagonal copy $\Delta\mathrm{U}(1)$ of $\mathrm{U}(1)$, which meets $\mathrm{SU}(2)$ in $\pm I$. We thus have diffeomorphisms $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\pm I\} \cong \mathrm{U}(2)/\Delta\mathrm{U}(1)$, and we consider the EMSS of $SO(3) \to B\Delta\mathrm{U}(1) \to B\mathrm{U}(2)$ over $\mathbb{F}_2$, beginning with $\mathrm{Tor}_{H^*B\mathrm{U}(2)}\big(\mathbb{F}_2, H^*\Delta\mathrm{U}(1)\big)$. To compute this, one can check the differential on $H^*B\Delta\mathrm{U}(1) \otimes \mathrm{U}(2) = \mathbb{F}_2[y_2] \otimes \Lambda[z_1, w_3]$ takes z_1 to 0 and w_3 to y_2^2, so the Tor is isomorphic to $\Lambda[z_1] \otimes \mathbb{F}_2[y_2]/(y_2^2)$. This is isomorphic to $H^*\mathrm{SO}(3) = \mathbb{F}_2[x_1]/(x_1^4)$ as a graded vector space but not as a ring, so the full multiplicative version of Theorem 0.2 is not true for $k = \mathbb{F}_2$.

7. Cup-i products

All later results showing EMSS collapse require notions of comparisons of $H^*(X)$ and $C^*(X)$ by linear maps that are multiplicative only up to homotopy. If $k[x, y]$ is the cohomology ring of some space X, finding an isomorphic copy of $k[x, y]$ in $C^*(X)$ itself is generally impossible, because lifting to representatives $x, y \in C^*(X)$, one only has $x \smile y \equiv (-1)^{|x||y|} y \smile x$ modulo a coboundary. This coboundary can be chosen in such a way as to yield a cochain homotopy from the cup product to $(x, y) \longmapsto (-1)^{|x||y|} y \smile x$, called the Steenrod ***cup-1 product*** and denoted $\smile_1$ [**Ste47**, §§2, 5]. The cup-1 product is itself commutative up to a cochain homotopy witnessed by an operation $\smile_2$, and inductively Steenrod found a sequence of binary operations $\smile_i$ of degree $-i$, each commutative up to a homotopy witnessed by $\smile_{i+1}$.

Definition 7.1. Given DGAs A and B, there is a natural DGA isomorphism $\chi_{A,B}\colon A \otimes B \longrightarrow B \otimes A$ given on pure homogeneous tensors by $a \otimes b \longmapsto (-1)^{|a||b|} b \otimes a$. A DGA A is said to ***admit cup-i products*** if for $0 \leqslant j \leqslant i$ there exist degree-$(-j)$ operations $\mu_j\colon A \otimes A \longrightarrow A$, starting with $\mu_0 = \mu_A$ the ring multiplication of A, such that for $0 \leqslant j < i$ one has $D\mu_{j+1} = \mu_j - \mu_j \chi_{A,A}$.

8. May

Peter May characterized the differentials of the algebraic EMSS in terms of generalized Massey products defined using matrices of cochains, called *matric Massey products*, and showed that in the EMSS associated to a bundle $F \to E \to B$, the elements of $C^*(B)$ figuring in these matric Massey products were iterated $\smile_1$-products of cocycles. He announced [**May68**] that he had found a DGA quasi-isomorphism $f\colon C^*(BT) \longrightarrow H^*(BT)$ for any k, which by the commutativity of $H^*(BT)$ annihilates $\smile_1$-products.[20] This induces a map $\mathrm{Tor}_{\mathrm{id}}(\mathrm{id}, f)$ from $\mathrm{Tor}_{C^*(BG)}(k, C^*BT)$ to $\mathrm{Tor}_{C^*(BG)}(k, H^*BT)$, which is an isomorphism by Proposition 5.4. The differentials in the EMSS converging to $\mathrm{Tor}_{C^*(BG)}(k, C^*BT)$ all involve $\smile_1$-products on $C^*(BT)$, which f annihilates, showing the algebraic EMSS converging to $\mathrm{Tor}_{C^*(BG)}(k, H^*BT)$ collapses at $E_2 = \mathrm{Tor}_{H^*(BG)}(k, H^*BT)$. With Baum's reduction, this establishes the following:

Theorem 8.1. *Let k be Noetherian. Then Theorem 0.2 holds additively up to an extension problem;* i.e., *with the filtration on $H^*(Y) \cong H^*\big(P_\bullet \otimes_{C^*(BG)} C^*(BK)\big)$*

---

[20] He also found DGA quasi-isomorphisms $C^*\big(\prod K(\pi_j, n_j); \mathbb{F}_2\big) \longrightarrow H^*\big(\prod K(\pi_j, n_j); \mathbb{F}_2\big)$ for positive integers n_j and finitely generated abelian groups π_j (with no 4-torsion for $n_j = 1$).

induced by the resolution degree p, the associated graded module $\operatorname{gr} H^*(Y)$ *is isomorphic to* $\operatorname{Tor}_{H^*(BG)}(k, H^*BK)$.

A preprint was circulated, but the proofs were involved and did not see print, suppressed in favor of later proofs joint with V.K.A.M Gugenheim.

9. Gugenheim–May

Gugenheim and May [**GM**] construct a DGA formality map $f\colon C^*(BT) \longrightarrow H^*(BT)$ annihilating $\smile_1$-products by dualizing DG Hopf algebra quasi-isomorphisms from homology $H_*(BT)$ to $C_*(BT)$, where the particular model of BT is a direct power of a simplicial model of BS^1. They assume only that k is a Noetherian ring. Then, using a variant of Baum's reduction 6.1 requiring a torsion hypothesis on BK, they reduce the EMSS collapse for Figure 5.1a to that for $G_T \to BT \xrightarrow{\rho} BG$.

As May had earlier observed, the map $\operatorname{Tor}_{\mathrm{id}}(\mathrm{id}, f)\colon \operatorname{Tor}_{C^*(BG)}(k, C^*BT) \to \operatorname{Tor}_{C^*(BG)}(k, H^*BT)$ is a linear isomorphism. To compute the codomain, Gugenheim–May resolve k as a $C^*(BG)$-module in the following way. Fix exterior generators $z_j \in H^*(G)$, corresponding polynomial generators $x_j = \tau z_j \in H^*(BG)$, and representatives $c_j \in C^*(BG)$. Equip the bigraded module $M = C^*(BG) \otimes H^*(G)$ with a differential d_M whose value on each pure tensor $1 \otimes z_{j_1} \cdots z_{j_n}$ differs from the naive choice $\sum_i \pm c_{j_i} \otimes z_{j_1} \cdots \widehat{z_{j_i}} \cdots z_{j_n}$ (the caret $\widehat{\ }$ denoting omission) by an element of the ideal generated by $\smile_1$-products on $C^*(BG)$.[21] Resolving k by M, one can compute $\operatorname{Tor}_{C^*(BG)}(k, H^*BT)$ from $H^*(BT) \otimes_{C^*(BG)} M \cong H^*(BT) \otimes H^*(G)$, where the differential annihilates $H^*(BT)$ and takes $1 \otimes \prod z_{j_i}$ to $(f\rho^* \otimes \mathrm{id}) d_M (1 \otimes \prod z_{j_i})$. But $f\rho^*$ annihilates the $\smile_1$-products distinguishing these values from those of the Cartan/Borel model, defined by $1 \otimes z_j \longrightarrow \rho^* x_j \otimes 1$, so we have recovered the classical $\operatorname{Tor}_{H^*(BG)}(k, H^*BT)$. This direct computation does not pass through an EMSS collapse result and thus also resolves the additive extension problem.

Theorem 9.1 (Gugenheim–May)**.** *Let k be a Noetherian ring such that $H^*(BK; \mathbb{Z})$ has no p-torsion for any factor p of* $\operatorname{char} k$ *(but is not necessarily polynomial). Then Theorem 0.2 holds additively.*

10. A-infinity notions historically

We have seen the assumption of a cup-one product on a DGA A limits how badly a map $k[x, y] \longrightarrow A$ lifting generators can fail to be a ring map. It is not hard to check a DGA A is commutative if and only if the multiplication $\mu\colon A \otimes A \longrightarrow A$, a cochain map, is in fact a DGA map, so a more systematic way of limiting noncommutativity is a system of homotopies moderating μ's failure to be multiplicative.

Notions of *strong homotopy multiplicativity* (SHM) and *strong homotopy commutativity* (SHC) originally apply to topological monoids and are due to Sugawara [**Su60**]. We cannot discuss them in detail here, but Sugawara was able to show that an SHM map between two monoids G and H induces a map $BG \longrightarrow BH$ of classifying spaces, that a countable CW-complex B has loop space ΩB SHC if and only if B is an H-space, and that for G a topological group, BG is an H-space if and only if G is SHC. Allan Clark algebraized these notions [**Cl65**][22] weakening Sugawara's hypotheses but obtaining similar consequences. He defines an SHM

[21] This is the best one can do, since $C^*(BG)$ is not actually commutative.

[22] He is interested in the monoid ΩX of variable-length (*Moore*) loops $\coprod_{r \geqslant 0} \operatorname{Map}\big(([0, r], \{0, r\}), (X, *)\big)$ on a pointed space X and particularly in comparing the

map of DGAs $A \longrightarrow A'$ as what is now called an ***A_∞-algebra map***, a sequence of maps $A^{\otimes n} \longrightarrow A'$ satisfying certain homotopy coherence conditions approximating multiplicativity. He shows these amount to a map of differential graded coalgebras $\mathbf{B}A \longrightarrow \mathbf{B}A'$, where $\mathbf{B}A$ is the *bar construction* defined in the next section.

Stasheff introduced associahedra K_n and A_n-spaces, and showed an A_n-space is equivalently a space X endowed with a sequence of maps $K_j \times X^j \longrightarrow X$ for $j \in [2, n]$ satisfying now–natural-seeming conditions [**St63a, St63b**]. He defined an A_n-map in such a way that when $n = \infty$, it is an SHM map in the sense of Sugawara. An ***A_n-algebra*** is an augmented graded module equipped with a sequence of linear maps $m_j\colon A^{\otimes j} \longrightarrow A$ for $j \in [1, n]$ satisfying formally similar conditions. Stasheff showed that an A_n-algebra is, in later language, a module for the n^{th} filtrand of the operad[23] of cellular chains on $K_\bullet$, so that if X is an A_n-space, $C_*(X)$ is an A_n-algebra. An ***A_∞-algebra*** is an augmented chain complex A which is an A_n-algebra for all n, amounting to a differential making the tensor coalgebra $\bigoplus_{p=0}^\infty \overline{A}^{\otimes p}$ a differential graded coalgebra. When A is a DGA, this prescription gives the differential on the bar construction of the next section, up to sign. Stasheff also connects the algebraic and topological bar constructions via a chain equivalence $\mathbf{B}C_*(G) \longrightarrow C_*(BG)$, for G a topological monoid, which will be used in Section 16.

11. The bar construction

The bar construction is a functor from DGA to a category DGC of cochain complexes satisfying axioms dual to the axioms for a ring with a derivation.

Definition 11.1. A ***coaugmented differential graded k-coalgebra*** (DGC) is a cochain complex (C, d_C) equipped with a ***comultiplication*** $\Delta_C\colon C \longrightarrow C \otimes C$ and maps $k \overset{\eta_C}{\to} C \overset{\varepsilon_C}{\to} k$ composing to id_k, respectively the ***coaugmentation*** and ***counit***, satisfying the identities

$$(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta, \qquad \Delta d = (d \otimes \mathrm{id} + \mathrm{id} \otimes d)\Delta, \qquad (\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta.$$

A (coaugmentation-preserving) DGC ***map*** is a degree-0 cochain map $g\colon C \longrightarrow C'$ satisfying $\Delta_{C'} g = (g \otimes g)\Delta_C$ and $\varepsilon'_C g = \varepsilon_C$ (and $g\eta_C = \eta'_C$). One writes $\Delta c = \sum c_{(1)} \otimes c_{(2)}$. A tensor product $C \otimes C'$ of DGCs becomes a DGC under the tensor differential and comultiplication taking $c \otimes c'$ to $\sum (-1)^{|c_{(2)}||c'_{(1)}|} c_{(1)} \otimes c'_{(1)} \otimes c_{(2)} \otimes c'_{(2)}$. When Δ is itself a DGC homomorphism $C \longrightarrow C \otimes C$, then C is called ***cocommutative***. Write $\Delta^{[n+1]} = (\Delta \otimes \mathrm{id}^{\otimes n-1})\Delta^{[n]}\colon C \longrightarrow C^{\otimes n}$ for the iterates of $\Delta^{[2]} = \Delta_C$ and $\overline{C} = \operatorname{coker} \eta_C$ for the coaugmentation coideal; a DGC C is ***cocomplete*** if each homogeneous element is annihilated by one of the composites $C \to C^{\otimes n} \to \overline{C}^{\otimes n}$. Write DGC for the category of cocomplete DGCs, and coaugmentation-preserving DGC maps. A DG ***Hopf algebra*** is a DGA A equipped with a DGA homomorphism $\Delta\colon A \longrightarrow A \otimes A$ (equivalently, a DGC A with a DGC map $\mu\colon A \otimes A \longrightarrow A$).

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monoids $\Omega(X \times X)$ and $\Omega X \times \Omega X$. These are homotopy equivalent but not homeomorphic, which had lead to an error in a lecture of Moore [**Mo60**, Thm. 7.II, pf.].

[23] We do not assume or use any specific results about operads in this survey, so we state this merely for historical context and those who already know about operads.

Example 11.2. The singular chain complex $C_*(X)$ associated to a topological space X becomes a DGC under the map taking a singular simplex $\sigma\colon \Delta^n \to X$ to the sum of $\sigma|_{\Delta[0,\ldots,p]} \otimes \sigma|_{\Delta[p,\ldots,n]} \in C_p(X) \otimes C_{n-p}(X)$ for $0 \leqslant p \leqslant n$.[24]

Using the Eilenberg–Zilber DGC quasi-isomorphism if X is an H-space, the composite DGC map $C_*(\mu_X) \circ \nabla$ makes $C_*(X)$ a DG Hopf algebra, and under sufficient flatness hypotheses, $H_*(X)$ becomes a cocommutative Hopf algebra. Dually, $C^*(X)$ becomes a DG Hopf algebra and $H^*(X)$ a commutative Hopf algebra.

Definition 11.3. The ***desuspension*** $s^{-1}\overline{A}$ of the submodule $\overline{A} = \ker \varepsilon_A$ of an augmented cochain complex A is $\overline{A}$ regraded via $(s^{-1}\overline{A})_n := \overline{A}_{n+1}$, given differential $d_{s^{-1}\overline{A}} = -s^{-1} d_A s$. On the direct sum $\mathbf{B}A$ of $\mathbf{B}_p A := (s^{-1}\overline{A})^{\otimes p}$ $(p \geqslant 0)$, the ***tensor coalgebra*** structure takes $s^{-1}a_1 \otimes \cdots \otimes s^{-1}a_p =: [a_1|\cdots|a_p] \in \mathbf{B}_p A$ to $\sum_{0 \leqslant \ell \leqslant p} [a_1|\cdots|a_\ell] \otimes [a_{\ell+1}|\cdots|a_p] \in \mathbf{B}A \otimes \mathbf{B}A$, where $[] = 1 \in k = \mathbf{B}_0 A$. When A is a DGA, the ***bar construction*** is the DGC structure on $\mathbf{B}A$ whose differential $d_{\mathbf{B}A}$ is the sum of the tensor differentials on the $\mathbf{B}_p A = (s^{-1}\overline{A})^{\otimes p}$ and the "bar-deletion" maps $\mathrm{id}^{\otimes i} \otimes s^{-1}\mu(s \otimes s) \otimes \mathrm{id}^{\otimes j}$ on $\mathbf{B}_{i+j+2}$ $(i, j \geqslant 0)$ taking $[a|b|c] \longmapsto \pm[a|bc] \pm [ab|c]$ and so on.[25]

We can use the bar construction to generalize the notion of commutativity. As we have noted, a DGA A is commutative if and only if $\mu\colon A \otimes A \longrightarrow A$ is itself a DGA map. Weakening this requirement, Stasheff–Halperin [**StaH70**, Def. 8] call a DGA A an ***SHC-algebra*** when it admits a DGC map $\Phi\colon \mathbf{B}(A \otimes A) \longrightarrow \mathbf{B}A$ such that $\Phi[a \otimes b] = [ab]$ for $a \otimes b \in \overline{A \otimes A}$—asking, in other words, only that μ be the unary component $(\overline{A \otimes A})^{\otimes 1} \longrightarrow A$ of an A_∞-algebra map.[26]

Definition 11.4 (See Husemoller *et al.* [**HMS74**, Def. IV.5.3])**.** There exists a natural transformation

$$\nabla\colon \mathbf{B}A_1 \otimes \mathbf{B}A_2 \longrightarrow \mathbf{B}(A_1 \otimes A_2)$$

of functors $\mathsf{DGA} \times \mathsf{DGA} \longrightarrow \mathsf{DGC}$, the ***shuffle map***, which is a homotopy equivalence of cochain complexes. It is the direct sum of the maps $\mathbf{B}_p A \otimes \mathbf{B}_q B \longrightarrow \mathbf{B}_{p+q}(A \otimes B)$ sending $[a_1|\cdots|a_p] \otimes [b_1|\cdots|b_q]$ to the sum of all tensor (p,q)-shuffles (with Koszul sign) of $[a_1 \otimes 1|\cdots|a_p \otimes 1|1 \otimes b_1|\cdots|1 \otimes b_q]$.

Clark notes that the composite $\Phi \circ \nabla$ makes $\mathbf{B}A$ a (possibly nonassociative) DG Hopf algebra. The earlier Eilenberg–Mac Lane paper [**EM53**, (7.7)] already showed this for $\Phi = \mathbf{B}\mu$ when A is a CDGA. We will strengthen this observation in Theorem 20.5.

The bar construction $\mathbf{B}A$ for a CDGA A was introduced in print by Eilenberg–Mac Lane [**EM53**, §11][27] not to parameterize homotopy-associative operations, but to provide a functorial resolution of a DG A-module M, proper projective when A, M, and $H^*(A)$ are all flat over k.

Definition 11.5. The ***one-sided bar construction*** of a DGA A and a right DG A-module M is the graded k-module $M \otimes \mathbf{B}A$ (note $M \otimes \mathbf{B}_0 A = M \otimes k = M$) equipped

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[24] This is the case $Y_\bullet = \operatorname{Sing} X$ of a more general DGC structure on the chain complex $C_*(Y_\bullet)$ associated to a simplicial set $Y_\bullet$.

[25] Signs are all determined by the Koszul convention $(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b)$.

[26] Clark's definition [**Cl65**] had asked even less, merely that $\Phi[a \otimes 1] = [a] = \Phi[1 \otimes a]$.

[27] This is the same work where the Eilenberg–Zilber map ∇ is introduced; the Eilenberg–Zilber theorem was originally proved using acyclic models. The bar shuffle $\nabla\colon \mathbf{B}A_1 \otimes \mathbf{B}A_2 \longrightarrow \mathbf{B}(A_1 \otimes A_2)$ and the observation it is a homotopy equivalence first appeared in Eilenberg–Mac Lane's sequel [**EM54**, Thm. 4.1a].

with the sum of the tensor differential and the maps $\ell_p\colon \mu_M(\mathrm{id}_M \otimes s)\otimes \mathrm{id}^{\otimes p-1}\colon M\otimes \mathbf{B}_pA \longrightarrow M\otimes\mathbf{B}_{p-1}A$ taking $m[a_1|\ldots|a_{p-1}|a_p] \longmapsto \pm m a_1[a_2|\ldots|a_p]$. Given a right A-module N, the ***two-sided bar construction*** is $(M\otimes\mathbf{B}A)\otimes_A N$, which can be identified with $M\otimes\mathbf{B}A\otimes N$ with the sum of the tensor differentials, the $\ell_p\otimes\mathrm{id}_N$, and the maps $r_p\colon -\mathrm{id}_M\otimes\mathrm{id}^{\otimes p-1}\otimes\mu_N(s\otimes\mathrm{id}_N)\colon M\otimes\mathbf{B}_pA\otimes N \longrightarrow M\otimes\mathbf{B}_{p-1}A\otimes N$ taking $m[a_1|\cdots|a_p]n \longmapsto \pm m[a_1|\cdots|a_{p-1}]a_p n$.

Mild flatness hypotheses imply $\mathrm{Tor}_A(M,N) = H^*\mathbf{B}(M,A,N)$.

12. Stasheff–Halperin

Stasheff–Halperin[28] [**StaH70**], in an offering to a workshop proceedings volume, suggested a program to generate collapse results for the EMSS of Figure 5.1a using an A_∞-map they construct. Given an SHC-algebra (A,Φ), define the ***iterates*** $\Phi^{[n]}\colon \mathbf{B}(A^{\otimes n}) \longrightarrow \mathbf{B}A$ of its structure map by $\Phi^{[2]} := \Phi$ and $\Phi^{[n+1]} := \Phi(\Phi^{[n]}\underline{\otimes}\,\mathrm{id}_A)$.[29] From a list $(f_j\colon A_j \to A)_{0\leqslant j<\kappa}$ of DGA maps, one can define a composite DGC map

$$\mathbf{B}\Big(\bigotimes A_j\Big) \xrightarrow[\mathbf{B}(\bigotimes f_j)]{} \mathbf{B}(A^{\otimes\kappa}) \xrightarrow[\Phi^{[\kappa]}]{} \mathbf{B}A \tag{12.1}$$

combining the f_j when κ is finite [**StaH70**, Thm. 9] (or, via a careful colimiting argument, countably infinite [**Mun74**, Prop. 3.9(i); §4.3]). We will call this map the ***compilation*** or *assembly* of the $\mathbf{B}f_j$.

When A is an SHC-algebra with countably generated polynomial cohomology $H^*A = k[x_1,x_2,\ldots] \cong \bigotimes_j k[x_j]$, a choice of representative $a_j \in A$ for each x_j induces a DGA map $f_j\colon k[x_j] \longrightarrow A$ sending x_j to a_j, so (12.1) gives a DGC map

$$\lambda_A\colon \mathbf{B}H^*A \longrightarrow \mathbf{B}A. \tag{12.2}$$

The map λ_A induces the identity map $H^*A = H^*(H^*A) \longrightarrow H^*A$ as follows.

Proposition 12.1 ([**Mun74**, Prop. 3.7])**.** *If A and B are* DGA*s and $g\colon \mathbf{B}A \to \mathbf{B}B$ a* DGC *map, the composite $\overline{A} \xrightarrow[s^{-1}]{} \mathbf{B}A \xrightarrow[g]{} \mathbf{B}B \twoheadrightarrow \overline{B}$ induces a graded algebra map*

$$H^*_\flat g\colon H^*A \longrightarrow H^*B$$

such that $H^*_\flat\lambda_A = \mathrm{id}_{H^*A}$.

Stasheff–Halperin describe an SHM *module* over a DGA A as a DG module M equipped with linear maps $A^{\otimes j}\otimes M \longrightarrow M$ whose adjoints $A^{\otimes j} \longrightarrow \mathrm{End}\,M$ yield an A_∞-map $\mathbf{B}A$ from $\mathbf{B}(\mathrm{End}\,M)$. Using this data they define a certain differential on $\mathbf{B}A\otimes M$ generalizing the one-sided bar construction of Definition 11.5 and reuse the notation $\mathrm{Tor}_A(k,M)$ for its cohomology. Then for λ_K and λ_G the $H^*_\flat$-quasi-isomorphisms of the previous paragraph and $\rho = B(K \hookrightarrow G)$, the composites

$$\begin{array}{ccccc} & \xrightarrow{\mathbf{B}H^*(\rho)} & \mathbf{B}H^*(BK) & \xrightarrow{\lambda_K} & \\ \mathbf{B}H^*(BG) & & & & \mathbf{B}C^*(BK) \\ & \xrightarrow[\lambda_G]{} & \mathbf{B}C^*(BG) & \xrightarrow[\mathbf{B}C^*(\rho)]{} & \end{array} \tag{12.3}$$

[28] written in this order; the pun in the title is also deliberate and similarly unexplained

[29] The iterates, including Φ, are themselves SHC-algebra maps [**Mun74**, Prop. 4.5].

induce two SHM $C^*(BG)$-structures on $\operatorname{Tor}_{C^*BG}(k, H^*BK)$. The maps λ_K and λ_G respectively induce graded isomorphisms

$$\operatorname{Tor}_{H^*BG}(k, H^*BK) \xrightarrow{\sim} \operatorname{Tor}^{\text{top}}_{H^*BG}(k, C^*BK),$$
$$\operatorname{Tor}^{\text{bottom}}_{H^*BG}(k, C^*BK) \xrightarrow{\sim} \operatorname{Tor}_{C^*BG}(k, C^*BK).$$

If the top and bottom SHM $C^*(BG)$-module structures on $H^*(BK)$ agreed, the composite isomorphism would imply Theorem 0.2 additively, but they do not, and later work (1) finds a homotopy between them or (2) postcomposes a map annihilating the difference. Both approaches involve the notion of a twisting cochain.

13. Twisting cochains

Twisting cochains were originally defined by Edgar Brown [**Br59**], who asked what algebraic information was necessary to generalize the Eilenberg–Zilber quasi-isomorphism $\nabla\colon C_*(B) \otimes C_*(F) \longrightarrow C_*(B \times F)$ from a product to a fiber bundle $F \to E \to B$. He found that $C_*(E)$ was quasi-isomorphic to the chain complex given by a new "twisted" differential on $C_*(B) \otimes C_*(F)$ defined using the information encoded in an inhomogeneous cochain in $C^*(B; C_*\Omega B)$, viewed as a map $t_B\colon C_*(B) \longrightarrow C_{*-1}(\Omega B)$, and the continuous action $\omega\colon \Omega B \times F \longrightarrow F$ via path-lifting. This new differential and the differential of the one-sided bar construction $\mathbf{B}A \otimes M$ are both instances of a single construction.

Definition 13.1 (Brown [**Br59**, §3])**.** For C a DGC and A a DGA, the ***cup product*** $f \smile g := \mu_A(f \otimes g)\Delta_C$ renders $\mathsf{Mod}(C, A)$ a DGA [**Mun74**, §1.8].

Proposition 13.2. *Let A be a* DGA*. The composite $t^A\colon \mathbf{B}A \twoheadrightarrow \mathbf{B}_1 A = s^{-1}\overline{A} \xrightarrow{\sim} \overline{A} \hookrightarrow A$ is a cochain map. For any* DGC *C,* DGC *maps $F\colon C \longrightarrow \mathbf{B}A$ correspond bijectively via $F \longmapsto t^A \circ F$ to maps $t \in \mathsf{Mod}_1(C, A)$ satisfying the three conditions*

$$\varepsilon_A t = 0 = t\eta_C, \qquad Dt = t \smile t. \tag{13.1}$$

Definition 13.3 (Brown [**Br59**, §3])**.** Let C be a DGC and A a DGA. An element of $\mathsf{Mod}_1(C, A)$ satisfying the conditions (13.1) is a ***twisting cochain***. We write $\mathsf{Tw}(C, A)$ for the set of these. The twisting cochain $t^A\colon \mathbf{B}A \longrightarrow A$, called the ***tautological twisting cochain***, is natural in the DGA A.

Definition 13.4 (Brown [**Br59**, §3])**.** Let C be a DGC, A a DGA, Q a differential right C-comodule, and N a differential left A-module. We define the ***cap product*** with an element $\phi \in \mathsf{Mod}(C, A)$ by

$$\delta^{\mathrm{R}}_\phi := (\mathrm{id}_Q \otimes \mu_N)\,(\mathrm{id}_Q \otimes \phi \otimes \mathrm{id}_N)\,(\Delta_Q \otimes \mathrm{id}_N)\colon Q \otimes N \longrightarrow Q \otimes N,$$
$$x \otimes y \longmapsto \sum \pm x_{(1)} \otimes \phi(x_{(2)})y.$$

When $\phi = t$ is a twisting cochain, $d_Q \otimes \mathrm{id}_N + \mathrm{id}_Q \otimes d_N - \delta^{\mathrm{R}}_t$ is a differential on $Q \otimes N$. The resulting cochain complex is the ***twisted tensor product*** $Q \otimes_t N$. Given another twisting cochain $t'\colon C \longrightarrow A'$, there is a similar "left" cap product $\delta^{\mathrm{L}}_{t'}$, for M a right DG module over a DGA A', and one can define a twisted tensor product $M \otimes_{t'} Q$ with differential differing from the tensor differential by $\delta^{\mathrm{L}}_{t'}$. Taking $Q = C$, one can also form a two-sided twisted tensor product $M \otimes_{t'} C \otimes_t N$.

Example 13.5. For A a DGA and M a right DG A-module, the one-sided bar construction of Section 11 is the twisted tensor product $M \otimes_{t^A} \mathbf{B}A$ and for a left DG A-module N, the two-sided bar construction $\mathbf{B}(M, A, N)$ is the two-sided twisted tensor product. More generally, for M an SHM module over A, the Stasheff–Halperin cochain complex defining their version of $\mathrm{Tor}_A(k, M)$ is computed by the twisted tensor product $\mathbf{B}A \otimes_t M$ for $t\colon \mathbf{B}A \to \mathbf{B}(\mathrm{End}\, M) \to \mathrm{End}\, M$. In Brown's setup, $C_*(B)$ is a DGC with comultiplication $\sigma \longmapsto \sum \sigma|_{[0,p]} \otimes \sigma|_{[p,|\sigma|]}$ and $C_*(F)$ is a $C_*(\Omega B)$-module via the composite $C_*(\Omega B) \otimes C_*(F) \xrightarrow[\nabla]{} C_*(\Omega B \times F) \xrightarrow[\omega_*]{} C_*(F)$.

14. Wolf

Joel Wolf's thesis work [W77, Thm. B] proves Theorem 0.2 for k a field. He replaces K with its maximal torus T *à la* Baum, then deals with the noncommutativity of (12.3) by replacing λ_T with a formality map $f\colon C^*(BT) \longrightarrow H^*(BT)$ of Gugenheim–May type, going in the opposite direction.

Write $\rho = B(T \hookrightarrow G)$. Then under our hypotheses the twisted tensor product $\mathbf{B}C^*(BG) \otimes_{t^{C^*(BT)}C^*(\rho)} C^*(BT)$ computes $\mathrm{Tor}_{C^*(BG)}(k, C^*BT) \cong H^*(G/T)$. Using maps of twisted tensor products induced by f and λ_G respectively, one gets

$$\mathrm{Tor}_{C^*(BG)}(k, C^*BT) \longrightarrow \mathrm{Tor}_{C^*(BG)}(k, H^*BT) \longleftarrow \mathrm{Tor}_{H^*(BG)}(k, H^*BT),$$

which a generalization of Proposition 5.4 shows are linear isomorphisms. The last Tor is computed by the twisted tensor product $\mathbf{B}H^*(BG) \otimes_t H^*(BT)$ with respect to $t = fC^*(\rho)t^{C^*(BG)}\lambda_G$ and Wolf will be done if he can show that t is equal to $H^*(\rho)t^{H^*(BG)}$ and hence gives the classical Tor.

Now $t^{H^*(BT)}\mathbf{B}\big(fC^*(\rho)\big)\lambda_G = fC^*(\rho)t^{C^*(BG)}\lambda_G$ by naturality of the tautological twisting cochain. Wolf shows one can select λ_G so that for $j \geqslant 2$, the images of the components $\lambda_j = t^{C^*(BG)} \circ \lambda_G|_{\mathbf{B}_j H^*(BG)}$ lie in the ideal generated by $\smile_1$-products. As $C^*(\rho)$ preserves $\smile_1$-products and f annihilates them, this means $fC^*(\rho)t^{C^*(BG)}\lambda$ vanishes on $\mathbf{B}_{\geqslant 2}H^*(BG)$. Since $H^*(\rho)t^{H^*(BG)}$ also vanishes on $\mathbf{B}_{\geqslant 2}H^*(BG)$ by definition, it remains only to check if the restrictions to $\mathbf{B}_1 H^*(BG) = s^{-1}\overline{H^*(BG)}$ agree. We can identify these with the two paths around the square

$$\begin{array}{ccccc} & & H^*(BT) & \xleftarrow{\ f\ } & \\ \overline{H^*(BG)} & \nearrow^{H^*(\rho)} \searrow_{\lambda_1 s} & & & C^*(BT) \\ & & C^*(BG) & \xrightarrow[C^*(\rho)]{} & \end{array} \tag{14.1}$$

By construction, we have identifications $H^*(\lambda_1 s) = H^*_\flat(\lambda_G) = \mathrm{id}_{\overline{H^*(BG)}}$ and $H^*(f) = \mathrm{id}_{H^*(BT)}$, so the maps in cohomology induced by $H^*(\rho)$ and $fC^*(\rho)\lambda_1 s$ are both $H^*(\rho)$. But $H^*(BG)$ and $H^*(BT)$ are DGAs with zero differential, so the maps induced in cohomology are the maps themselves, meaning $fC^*(\rho)\lambda_1 s$ and $H^*(\rho)$ are themselves equal, completing the proof.

Theorem 14.1 (Wolf)**.** *Let k be a field. Then Theorem 0.2 holds additively, even without the hypothesis on $H^*(BK)$.*

15. The cobar construction

The work of Husemoller–Stasheff–Moore [HMS74] completing the program proposed in Stasheff–Halperin provides a wholesale reformulation of differential homological algebra that allows them to reprove many of the known quasi-isomorphisms. As this approach is homological, we will need to dualize several notions.

Proposition 15.1. *Given a* DGC C*, there exists a twisting cochain* $t_C\colon C \longrightarrow \mathbf{\Omega}C$ initial *in the sense that for any* DGA A*, any twisting cochain* $t\colon C \longrightarrow A$ *factors uniquely through a* DGA *map* $f^t\colon \mathbf{\Omega}C \longrightarrow A$ *with* $t = f^t \circ t_C$*.*

Definition 15.2. The DGA $\mathbf{\Omega}C$ is referred to as the ***cobar construction*** and gives the object component of a functor $\mathbf{\Omega}\colon \mathsf{DGC} \longrightarrow \mathsf{DGA}$ [**Mun74**, §1.7]. The tautological twisting cochain $t_{(-)}\colon \mathrm{id} \longrightarrow \mathbf{\Omega}$ is a natural transformation. As an algebra, the cobar construction $\mathbf{\Omega}C$ is the tensor algebra on the suspension $s\overline{C}$ of the coaugmentation coideal, and the differential is the sum of all the differentials $\mathrm{id}^{\otimes i} \otimes d_{s\overline{C}} \otimes \mathrm{id}^{\otimes j}$ and operations $\mathrm{id}^{\otimes i} \otimes (s \otimes s)\Delta_C s^{-1} \otimes \mathrm{id}^{\otimes j}$. There is also a dual version of the shuffle map of Definition 11.4, a natural transformation

$$\gamma\colon \mathbf{\Omega}(C_1 \otimes C_2) \longrightarrow \mathbf{\Omega}C_1 \otimes \mathbf{\Omega}C_2$$

of functors $\mathsf{DGC} \times \mathsf{DGC} \longrightarrow \mathsf{DGA}$.

The two functors $\mathbf{\Omega} \dashv \mathbf{B}$ form an adjoint pair [**Mun74**, §1.9–10]:

$$\frac{g_t\colon \mathbf{\Omega}C \longrightarrow A}{f^t\colon C \longrightarrow \mathbf{B}A,}$$

linked by the twisting cochain $t\colon C \longrightarrow A$. We will have frequent recourse to the unit and counit of the adjunction $\mathbf{\Omega} \dashv \mathbf{B}$,

$$\eta\colon \mathrm{id} \longrightarrow \mathbf{B\Omega} \qquad \text{and} \qquad \varepsilon\colon \mathbf{\Omega B} \longrightarrow \mathrm{id}$$

respectively. These are both natural quasi-isomorphisms and homotopy equivalences on the level of DG modules [**HMS74**, Thm. II.4.4–5][**Mun74**, Cor. 2.15].

Definition 15.3. Dualizing the diagrams defining a DG module over a DGA A gives a notion of a ***DG comodule*** over a DGA C. Given a right DG A-module M and a left DG A-module N, the tensor product $M \otimes_A N$ can be understood as the kernel of the map $\mu_M \otimes \mathrm{id}_N - \mathrm{id}_M \otimes \mu_N\colon M \otimes A \otimes N \longrightarrow M \otimes N$, and dually, given a left DG C-comodule M and a right DG C-comodule N, one can define a ***cotensor product*** $M \mathbin{\square_C} N$ as the cokernel of $\Delta_M \otimes \mathrm{id}_N - \mathrm{id}_M \otimes \Delta_N\colon M \otimes N \longrightarrow M \otimes C \otimes N$. A notion of ***proper injective resolution*** $I_\bullet$ of a DG C-comodule M is defined dually to the notion of proper projective resolution and $\mathrm{Cotor}_C(M,N)$ is defined as $H_*(I_\bullet \mathbin{\square_C} N)$. Subject to k-flatness of C, $H_*(C)$, N, and $H_*(N)$, there is a homological algebraic EMSS $\mathrm{Cotor}^{H_*(C)}(H_*M, H_*N) \implies \mathrm{Cotor}_C(M,N)$, which can be computed using a one-sided cobar construction $M \otimes \mathbf{\Omega}C$ as a proper injective resolution of M.

There is also a homological Eilenberg–Moore theorem, stating that the square of Figure 5.1c induces a coalgebra isomorphism $\mathrm{Cotor}^{C_*(B)}(C_*X, C_*E) \cong H_*(Y)$ (reducing to Adams's theorem from Remark 15.4 when $X \simeq * \simeq E$, so that $Y \simeq \Omega B$), and there corresponds a homological EMSS $\mathrm{Cotor}^{H_*(B)}(H_*X, H_*E) \implies H_*(Y)$.

Remark 15.4. Adams [**A56**] introduced the cobar construction as a model for the loop space ΩB of a simply-connected space B and proved $H_*\mathbf{\Omega}C_*(B) \cong H_*(\Omega B)$ (we dually have $H^*\mathbf{B}C^*(B) \cong H^*(\Omega B)$ via Eilenberg–Moore).[30] He was motivated by a desire to rephrase his joint result with Hilton [**AH56**], which gives a DGA model for the Pontrjagin ring $H_*(\Omega B)$ in terms of the DGC $C_*(B)$, in terms of a functor $\mathbf{\Omega}\colon \mathsf{Mod} \longrightarrow \mathsf{Mod}$ which would in principle be iterable.

[30] Rivera and Zeinalian recently extended this to path-connected spaces [**RZ18**, **R19**].

Baues [**Baues81**] found a comultiplication on $\mathbf{\Omega} C_*(B)$ making it a DG Hopf algebra and inducing the standard comultiplication on $H_*(\Omega B)$, and allowing one iteration.[31] Dually, he found a multiplication on $\mathbf{B}C^*(B)$ making it a DG Hopf algebra and inducing the cup product on $H^*(\Omega B)$. This multiplication in fact renders $C^*(B)$ a *homotopy Gerstenhaber algebra* in the sense we will discuss in Section 20.

16. Husemoller–Stasheff–Moore

Husemoller–Moore–Stasheff's proof of Theorem 0.2 first reduces to $K = T$ a torus following Baum, then proves the collapse of the homological EMSS of Figure 5.1a as follows. Taking the canonical simplicial model $K(\mathbb{Z}^n, 1)$ for T, they note $C_*(T)$ is a commutative DG Hopf algebra and there are quasi-isomorphisms $H_*(BT) \cong H_*\mathbf{B}C_*(T) \longrightarrow \mathbf{B}C_*(T) \cong C_*(BT)$ of divided power Hopf algebras. Moreover, they note that the cobar construction of both domain and codomain carry a cocommutative DG Hopf algebra structure Δ^* preserved by this map [**HMS74**, Prop. IV.6.1]. If k is chosen such that $H_*(G)$ is an exterior Hopf algebra on exterior generators x_j, there are induced DGC maps $C_*(BG) \xrightarrow{\sim} \mathbf{B}C_*G \to \mathbf{B}\Lambda[x_j]$, whose adjoint DGA maps $\mathbf{\Omega} C_*(BG) \longrightarrow \Lambda[x_j]$ assemble to a map

$$\mathbf{\Omega B}C_*(G) \xrightarrow{\sim} \mathbf{\Omega}C_*(BG) \xrightarrow{(\Delta^*)^{[\mathrm{rk}\,G]}} \big(\mathbf{\Omega}C_*(BG)\big)^{\otimes\,\mathrm{rk}\,G} \longrightarrow \bigotimes \Lambda[x_j] \cong H_*(G)$$

with adjoint $\mathbf{B}C_*(G) \longrightarrow \mathbf{B}H_*(G)$ a quasi-isomorphism. They show the composite

$$H_*(BT) \longrightarrow C_*(BT) \longrightarrow C_*(BG) \longrightarrow \mathbf{B}H_*(G)$$

factors through

$$H_*(BG) \xrightarrow{\sim} \bigotimes \mathbf{B}\Lambda[x_j] \xrightarrow{\nabla^{[\mathrm{rk}\,n]}} \mathbf{B}\big(\bigotimes \Lambda[x_j]\big) \xrightarrow{\sim} \mathbf{B}H_*(G)$$

because $H_*(BT) = H_*\mathbf{B}C_*(T)$ is commutative [**HMS74**, Prop. IV.7.2], and then the commutativity of the diagram

$$\begin{array}{ccc} H_*(BT) & \longrightarrow & H_*(BG) \\ \Big\downarrow & & \downarrow \\ & & \mathbf{B}H_*(G) \\ & & \uparrow \\ C_*(BT) & \longrightarrow & C_*(BG) \end{array}$$

gives the desired sequence of quasi-isomorphisms

$$\begin{array}{ccc} \mathrm{Cotor}^{H_*(BG)}\big(k, H_*(BT)\big) & \xrightarrow{\sim} & \mathrm{Cotor}^{\mathbf{B}H_*(G)}\big(k, H_*(BT)\big) \\ & & \uparrow\wr \\ .\mathrm{Cotor}^{C_*(BG)}\big(k, C_*(BT)\big) & \xleftarrow[\sim]{} & \mathrm{Cotor}^{C_*(BG)}\big(k, H_*(BT)\big) \end{array}$$

The DG Hopf algebra structure can be seen as a strategy for compilation of maps replacing the Stasheff–Halperin (12.1).

After all the work poured into this long paper, there is a minor hitch in the proof of a lemma near the very end. This lemma, however, can be substituted with a result of the present author, proven for other reasons, and an alternative

[31] But this goes no further: there is no suitable diagonal on $\mathbf{\Omega\Omega}C_*(B)$.

definition due to Munkholm (see Theorem 17.5) of the map ψ they use in defining the coproducts Δ^* on $\mathbf{\Omega}C_*(BG)$ and $\mathbf{\Omega}C_*(BT)$.[32]

Theorem 16.2. *Let k be a ring such that $H_*(G)$ and $H_*(K)$ are exterior algebras on odd-degree generators. Then there is an isomorphism of graded modules*

$$H_*(G/K) \cong \mathrm{Cotor}^{H_*(BG)}\big(k, H_*(BK)\big).$$

17. Munkholm

Munkholm proves Theorem 0.3 by showing (12.3) is homotopy-commutative in a strong sense, assuming only that k is a principal ideal domain and a certain extra condition in characteristic 2.

Definition 17.1 ([**Mun74**, §1.11][**Mun78**, §4.1])**.** Given DGA maps $f_0, f_1 \colon A' \longrightarrow A$, a DGA ***homotopy*** $f_0 \simeq f_1$ is a map $h \colon A' \longrightarrow A$ of degree -1 such that

$$\varepsilon_A h = 0, \qquad h\eta_{A'} = 0, \qquad d(h) = f_0 - f_1, \qquad h\mu_{A'} = \mu_A(f_0 \otimes h + h \otimes f_1)\ .$$

Given twisting cochains $t_0, t_1 \colon C \longrightarrow A$, a ***twisting cochain homotopy*** $t_0 \simeq t_1$ is a map $x \colon C \longrightarrow A$ of degree 0 such that

$$\varepsilon_A x = \varepsilon_C, \qquad x\eta_A = \eta_C, \qquad d(x) = t_0 \smile x - x \smile t_1.$$

Given DGC maps $g_0, g_1 \colon C \longrightarrow C'$, a DGC ***homotopy*** $g_0 \simeq g_1$ is a map $j \colon C \longrightarrow C'$ of degree -1 such that

$$\varepsilon_{C'} j = 0, \qquad j\eta_C = 0, \qquad d(j) = g_1 - g_0, \qquad \Delta_{C'} j = (g_0 \otimes j + j \otimes g_1)\Delta_C.$$

The bijections

$$\mathsf{DGA}(\mathbf{\Omega}C, A) \cong \mathsf{Tw}(C, A) \cong \mathsf{DGC}(C, \mathbf{B}A)$$

preserve these notions of homotopy, as do the functors $\mathbf{\Omega}$ and $\mathbf{B}$.

Munkholm enhances the notion of strong homotopy commutativity by asking that the structure map $\Phi \colon \mathbf{B}(A \otimes A) \longrightarrow \mathbf{B}A$ satisfy up-to-homotopy unitality, commutativity, and associativity criteria. The canonical example is that of an authentically commutative algebra.

Example 17.2. If A is a CDGA, then $\Phi_A \coloneqq \mathbf{B}\mu_A \colon \mathbf{B}(A \otimes A) \longrightarrow \mathbf{B}A$ makes A an SHC-algebra. The cohomology ring $H^*(X_\bullet; k)$ of a simplicial set is of this type, and will always be considered with this SHC-algebra structure. If $\rho \colon A \longrightarrow B$ is a map of CDGAs, then $\mathbf{B}\rho$ is an SHC-algebra map per the coming Definition 17.7.

Example 17.3. If A is an SHC-algebra, then $H^*(A)$ is a CGA, so $\Phi_{H^*(A)} \coloneqq \mathbf{B}\mu_{H^*(A)}$ gives an SHC-algebra structure on $H^*(A)$ by Example 17.2. The cohomology ring of an SHC-algebra will always be endowed with this SHC-algebra structure.

---

[32] In the proof of Proposition IV.6.1, the commutativity of the necessary diagram depends on Proposition IV.5.7, which one can check by hand is false. However, this diagram can be replaced by another relevant diagram, which actually commutes owing to the following result:

Lemma 16.1 (Carlson (unpublished))**.** *Let A_1 and A_2 be DGAs, ∇ as given in Definition 11.4, and ψ as in Theorem 17.5. Then the map γ of Definition 15.2 is equal to the composition*

$$\mathbf{\Omega}(\mathbf{B}A_1 \otimes \mathbf{B}A_2) \xrightarrow{\mathbf{\Omega}\nabla} \mathbf{\Omega}\mathbf{B}(A_1 \otimes A_2) \xrightarrow{\psi} \mathbf{\Omega}\mathbf{B}A_1 \otimes \mathbf{\Omega}\mathbf{B}A_2.$$

Theorem 17.4 ([Mun74, Prop. 4.7]). *Let X be a simplicial set and k any ring. Then the normalized cochain algebra $C^*(X) = C^*(X;k)$ admits a natural* SHC-*algebra structure $\Phi_{C^*(X)}$.*[33]

Munkholm defines an SHC-algebra map to be a DGC map making a natural square commute, but to define the square, we require some auxiliary concepts.

Theorem 17.5 ([HMS74, Prop. IV.5.5] [Mun74, k_{A_1,A_2}, p. 21, via Prop. 2.14]). *There exists a homotopy equivalence of cochain complexes*

$$\psi\colon \mathbf{\Omega B}(A_1 \otimes A_2) \longrightarrow \mathbf{\Omega B}A_1 \otimes \mathbf{\Omega B}A_2$$

which is a natural transformation of functors $\mathsf{DGA} \times \mathsf{DGA} \longrightarrow \mathsf{DGA}$ *satisfying*

$$(\varepsilon_{A_1} \otimes \varepsilon_{A_2}) \circ \psi = \varepsilon_{A_1 \otimes A_2}\colon \mathbf{\Omega B}(A_1 \otimes A_2) \longrightarrow A_1 \otimes A_2$$

and reducing to the identity if A_1 or A_2 is k.

Definition 17.6 ([Mun74, Prop. 3.3]). Let A_j, B_j be DGAs and $g_j\colon \mathbf{B}A_j \to \mathbf{B}B_j$ DGC maps for $j \in \{1,2\}$. The ***internal tensor product*** $g_1 \underline{\otimes} g_2$ is the composite

$$\mathbf{B}(A_1 \otimes A_2) \xrightarrow{\eta} \mathbf{B\Omega B}(A_1 \otimes A_2) \xrightarrow{\mathbf{B}\psi} \mathbf{B}(\mathbf{\Omega B}A_1 \otimes \mathbf{\Omega B}A_2) \xrightarrow{\mathbf{B}(\varepsilon\mathbf{\Omega}g_1 \otimes \varepsilon\mathbf{\Omega}g_2)} \mathbf{B}(B_1 \otimes B_2).$$

This construction becomes functorial on passing to the homotopy category, or if enough of the DGC maps g_j involved are $\mathbf{B}f_j$ for DGA maps f_j [Mun74, Prop. 3.3(ii)]. It is related to the ordinary tensor product by the shuffle map in the sense that $\nabla \circ (g_1 \otimes g_2) = (g_1 \underline{\otimes} g_2) \circ \nabla\colon \mathbf{B}A_1 \otimes \mathbf{B}A_2 \longrightarrow \mathbf{B}(B_1 \otimes \mathbf{B}_2)$ [Fr21a, Lem. 4.4].

Definition 17.7. Given SHC-algebras Z and A, a DGC map $g\colon \mathbf{B}Z \longrightarrow \mathbf{B}A$ is said to be an ***SHC-algebra map*** if there exists a DGC homotopy between the two paths around the square

$$\begin{array}{ccc} \mathbf{B}(Z \otimes Z) & \xrightarrow{\Phi_Z} & \mathbf{B}Z \\ {\scriptstyle g \underline{\otimes} g}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbf{B}(A \otimes A) & \xrightarrow[\Phi_A]{} & \mathbf{B}A. \end{array} \tag{17.1}$$

Munkholm also generalizes the construction of (12.1): given a finite or countable list $(g_j\colon \mathbf{B}A_j \longrightarrow \mathbf{B}A)_{0 \leqslant j < \kappa}$ of DGC maps, the ***compilation***

$$\mathbf{B}\big(\bigotimes A_j\big) \xrightarrow[\underline{\bigotimes} g_j]{} \mathbf{B}(A^{\otimes\kappa}) \xrightarrow[\Phi^{[\kappa]}]{} \mathbf{B}A \tag{17.2}$$

is again a DGC map. This operation enjoys good homotopy properties.

Proposition 17.8 ([Mun74, Props. 3.9(iv) & 4.6]). *The* DGC *homotopy class of the compiled map* (17.2) *depends only on the homotopy classes of the inputs g_j.*

Proposition 17.9 ([Mun74, p. 44, top]). *Up to* DGC *homotopy, postcomposition with an* SHC-*algebra map commutes with compilation.*

Using this enhanced notion, Munkholm is also able to verify homotopies by checking them on generators.

[33] This natural SHC structure on cochains is a reinterpretation of the classical Eilenberg–Zilber theorem; only verifying the homotopy-associativity axiom requires much additional work.

Theorem 17.10 ([**Mun74**, Lem. 7.3])**.** *If A is an* SHC*-algebra with polynomial cohomology, the* DGC *map $\lambda_A\colon \mathbf{B}H^*A \longrightarrow \mathbf{B}A$ of* (12.2) *is an* SHC*-algebra map when*

- *the characteristic of k is not 2, or*
- *the characteristic of k is 2, and the cup-one–squares $x_j \smile_1 x_j$ of the chosen generators x_j of H^*A are zero.*

Proof. Recall that λ_A is the compilation of maps $\mathbf{B}\big(f_j\colon k[x_j] \to A\big)$ where $f_j(x_j) = a_j \in A$ represents $x_j \in H^*(A)$. One checks that $\lambda_A \mathbin{\underline{\otimes}} \lambda_A$ is DGC-homotopic to the compilation of the maps $\mathbf{B}\big(f_j^{\otimes 2}\colon k[x_j]^{\otimes 2} \to A^{\otimes 2}\big)$, so by Proposition 17.8, λ_A will be an SHC-algebra map if and only if each $\mathbf{B}f_j$ is. To check this, form the associated square (17.1), postcompose t^A and check if the twisting cochains $t' = t^A\Phi_A\mathbf{B}(f_j \otimes f_j)$ and $t'' = t^A\mathbf{B}(f_j\mu_{k[x_j]}) = f_j\mu_{k[x_j]}t^{k[x_j]^{\otimes 2}}$ are homotopic. Write $x = 1 \otimes x_j$ and $y = x_j \otimes 1$. Munkholm [**Mun74**, Prop. 6.2] characterizes homotopy classes of twisting cochains $t\colon \mathbf{B}k[x,y] \longrightarrow A$ in terms of the classes in $H^*(A)$ of three cochains: $t[x]$, $t[y]$, and a certain cocycle $c_{12}(t)$ of degree $|x|+|y|-1$. One easily checks $t'(x) = t''(x) = a_j = t''(y) = t'(y)$. If $\operatorname{char} k \neq 2$, then $H^*(A)$ is evenly graded, so $c_{12}(t') = 0 = c_{12}(t'')$ for degree reasons. If $\operatorname{char} k = 2$, it turns out $c_{12}(t') = 2a_j \smile_1 a_j = 0$ and $c_{12}(t'') = a_j \smile_1 a_j$, so t' and t'' are homotopic if and only if the class in $H^{2|a_j|-1}(A)$ of $a_j \smile_1 a_j$ is zero. □

Theorem 17.11 ([**Mun74**, §7.4])**.** *Given* SHC*-algebras A, X,* SHC*-algebra maps $\lambda_X\colon\ \mathbf{B}H^*X \longrightarrow \mathbf{B}X$ and $\xi\colon \mathbf{B}A \longrightarrow \mathbf{B}X$, if H^*A is a polynomial ring on countably many generators, λ_A is the* DGC *map of* (12.2)*, and $H^*_\flat\lambda_X$ is the identity, then there exists a* DGC *homotopy between the two paths around the central square below:*

$$\mathbf{B}k[x_j] \xrightarrow{\mathbf{B}i_j} \mathbf{B}H^*(A) \rightrightarrows^{\mathbf{B}H^*_\flat\xi\ \to\ \mathbf{B}H^*(X)\ \xrightarrow{\lambda_X}}_{\lambda_A\ \to\ \mathbf{B}A\ \xrightarrow{\xi}} \mathbf{B}X \xrightarrow{t^X} X, \tag{17.3}$$

where $H^*(A) = k[x_1,x_2,x_3,\ldots]$ and $i_j\colon k[x_j] \longhookrightarrow H^*(A)$ are the inclusions.

Proof. First we examine the twisting cochain homotopy classes of the two composites $t\colon \mathbf{B}k[x_j] \to X$ in (17.3) for each j. Each is determined [**Mun74**, Prop. 6.2] by the class of $t[x_j]$ in $H^*(X)$, and because $H^*_\flat(\mathbf{B}H^*_\flat\xi) = H^*_\flat\xi$ and $H^*_\flat\lambda_A$ and $H^*_\flat\lambda_X$ are identity maps, on unravelling definitions one sees they are both $(H^*_\flat\xi)(x_j)$. By Definition 17.1, this implies DGC homotopies between the composites $\mathbf{B}k[x_j] \to \mathbf{B}X$. By Proposition 17.8, it will suffice to see that $\lambda_X \circ \mathbf{B}(H^*_\flat\xi)$ is DGC-homotopic to the compilation of the $\lambda_X \circ \mathbf{B}(H^*_\flat\xi \circ i_j)$ and $\xi \circ \lambda_A$ to the compilation of the $\xi \circ \lambda_A \circ \mathbf{B}i_j$. For the former, recall that λ_X and $\mathbf{B}H^*_\flat\xi$ are SHC-algebra maps while $\mathrm{id}_{\mathbf{B}H^*(A)} = \mathbf{B}(\bigotimes i_j)$ is the compilation of the $\mathbf{B}i_j$, and use Proposition 17.9. For the latter, note ξ is an SHC-algebra map and λ_A was defined as the compilation $\Phi_A^{[\kappa]} \circ \mathbf{B}(\bigotimes f_j)$ of the $\mathbf{B}f_j$ for $f_j\colon k[x_j] \to A$, so precomposing $\mathbf{B}i_j$ recovers $\mathbf{B}f_j$. □

By Theorem 17.4, cochain algebras are SHC-algebras, so in the situation of the Eilenberg–Moore theorem 5.2, if $X \xrightarrow{\chi} B \xleftarrow{\pi} E$ have countably-generated polynomial cohomology, one can apply (12.2) three times to obtain the following:

$$
\begin{array}{ccccc}
\mathbf{B}H^*(X) & \xleftarrow{\mathbf{B}H^*(\chi)} & \mathbf{B}H^*(B) & \xrightarrow{\mathbf{B}H^*(\pi)} & \mathbf{B}H^*(E) \\
\downarrow{\scriptstyle \lambda_X} & & \downarrow{\scriptstyle \lambda_B} & & \downarrow{\scriptstyle \lambda_E} \\
\mathbf{B}C^*(X) & \xleftarrow[\mathbf{B}C^*(\chi)]{} & \mathbf{B}C^*(B) & \xrightarrow[\mathbf{B}C^*(\pi)]{} & \mathbf{B}C^*(E).
\end{array}
\tag{17.4}
$$

If X and E satisfy the additional hypotheses one can apply Theorem 17.10, then λ_X and λ_E are SHC-maps, so applying Theorem 17.11 twice to (17.4), both squares commute via DGC homotopies. Applying $\mathbf{\Omega}$ to (17.4) gives a diagram $\mathbf{\Omega}$(17.4) of DGAs commutative up to DGA homotopy. We want to see this induces a linear isomorphism $\mathrm{Tor}_{H^*(B)}(H^*X, H^*E) \xrightarrow{\sim} \mathrm{Tor}_{C^*(B)}(C^*X, C^*E)$. The DGAs are all of the wrong form, $\mathbf{\Omega B}A$ but the counit $\mathbf{\Omega B}A \longrightarrow A$ of the $\mathbf{\Omega} \dashv \mathbf{B}$ adjunction is a natural DGA quasi-isomorphism, so by two applications of Proposition 5.4, we can instead show $\mathbf{\Omega}$(17.4) induces an isomorphism $\mathrm{Tor}_{\mathbf{\Omega B}H^*(B)}(\mathbf{\Omega B}H^*X, \mathbf{\Omega B}H^*E) \xrightarrow{\sim} \mathrm{Tor}_{\mathbf{\Omega B}C^*(B)}(\mathbf{\Omega B}C^*X, \mathbf{\Omega B}C^*E)$.

Munkholm achieves this by re-encoding homotopies. Recall that a homotopy $j\colon g_0 \simeq g_1\colon C \longrightarrow C'$ of maps of chain complexes is equivalent by a single map $C \otimes I \longrightarrow C'$, where I is the complex $k\{u_{[0,1]}\} \to k\{u_{[0]}, u_{[1]}\}$ of nondegenerate chains in the standard simplicial structure on the interval $[0,1]$. For DGAs, the dual algebra $I^* = k\{v_0, v_1, e\}$ of normalized cochains on the simplicial interval plays an analogous role [**Mun74**, Thm. 5.4, pf.]. For any DGA A, the DGA $I^* \otimes A$ comes with two (non-unital) DGA maps $\pi_j\colon I^* \otimes A \twoheadrightarrow kv_j \otimes A \xrightarrow{\sim} A$, and a DGA homotopy $h\colon f_0 \simeq f_1\colon A' \longrightarrow A$ induces a DGA map $h^P\colon A' \longrightarrow I^* \otimes A$ with $f_j = \pi_j \circ h^P$.[34] Since $H^*(I^*) \cong k$, one sees the π_j are quasi-isomorphisms.

Now $\mathbf{\Omega}$ from (17.4) induces a map of Tors in the following way [**Mun74**, Thm. 5.4]. Given DGA maps as in (5.2), with the squares commuting up to DGA homotopies represented by $h^P_M\colon A' \longrightarrow I^* \otimes M'$ and $h^P_N\colon A' \longrightarrow I^* \otimes N$. Then the following diagram commutes by definition:

$$
\begin{array}{ccccccc}
M' & \xrightarrow{u} & M & \xleftarrow{\pi_0} & I^* \otimes M & \xrightarrow{\pi_1} & M \\
\uparrow{\scriptstyle \phi_{M'}} & & \uparrow & & \uparrow{\scriptstyle h^P_M} & & \uparrow{\scriptstyle \phi_M} \\
A' & = & A' & = & A' & \xrightarrow{f} & A \\
\downarrow{\scriptstyle \phi_{N'}} & & \downarrow & & \downarrow{\scriptstyle h^P_N} & & \downarrow{\scriptstyle \phi_N} \\
N' & \xrightarrow[v]{} & N & \xleftarrow[\pi_0]{} & I^* \otimes N & \xrightarrow[\pi_1]{} & N,
\end{array}
\tag{17.5}
$$

inducing a graded linear map $\mathrm{Tor}_f(\pi_1, \pi_1) \circ \mathrm{Tor}_{\mathrm{id}}(\pi_0, \pi_0)^{-1} \circ \mathrm{Tor}_{\mathrm{id}}(u, v)$, where $\mathrm{Tor}_{\mathrm{id}}(\pi_0, \pi_0)$ is an isomorphism by Proposition 5.4. If u, f, and v are also quasi-isomorphisms, the composite is similarly an isomorphism.

Applying this to $\mathbf{\Omega}$(17.4), we get Munkholm's main result.

[34] To render these maps unital and augmentation-preserving, which is important for any further use of the adjunction, replace $I^* \otimes A$ with the sub-DGA $PA = k\{(v_0 + v_1) \otimes 1\} \oplus I^* \otimes \overline{A}$ and modify the π_j in the obvious way; then DGA maps $A' \to PA$ and DGA homotopies of DGA maps $A' \to A$ are in bijection.

Theorem 17.12. *In the situation of Theorem 5.2, suppose that $H^*(X)$, $H^*(B)$, and $H^*(E)$ are polynomial rings on at most countably many generators, and if* $\operatorname{char} k = 2$, *assume there exist polynomial generators for $H^*(X)$ and $H^*(E)$ whose $\smile_1$-squares vanish. Then there is a graded linear isomorphism*

$$\operatorname{Tor}_{H^*(B)}(H^*X, H^*E) \xrightarrow{\sim} H^*(Y).$$

Corollary 17.13. *Let k be a principal ideal domain. Then Theorem 0.2 holds additively.*

Munkholm observes that applying Baum's reduction 6.1 and his own corollary 17.13 for $K = T$ a torus, one recovers the Gugenheim–May theorem 9.1.

Counterexample 17.14. One might hope the isomorphism of Theorem 17.12 is multiplicative, but it is not without some added conditions. Let B be the Eilenberg–Mac Lane space $K(\mathbb{Z}/2, 2)$ and E the contractible space PB of paths in B starting at a fixed basepoint, with $PB \to B$ evaluation at the other end. If we take for X the basepoint of B, the pullback Y is the loop space $\Omega B = K(\mathbb{Z}/2, 1) \simeq \mathbb{R}\mathrm{P}^\infty$, with polynomial cohomology ring $\mathbb{F}_2[\iota_1]$ over $k = \mathbb{F}_2$. On the other hand, $H^*K(\mathbb{Z}/2, 2)$ is a polynomial ring on generators of degrees $2, 3, 5, 9, 17, \ldots$, so that $\operatorname{Tor}_{H^*K(\mathbb{Z}/2,2)}(\mathbb{F}_2, \mathbb{F}_2)$ is exterior on generators of degrees $1, 2, 4, 8, 16, \ldots$. Theorem 17.12 applies and correctly reflects that the underlying graded $\mathbb{F}_2$-modules are isomorphic, but this isomorphism is not multiplicative.

18. Singhof

Singhof [**Si93**] directly applies Munkholm's theorem 17.12 to Figure 5.1b in the case $H \times K$ acts freely on G, so that $H\backslash G/K$ is a smooth manifold.

Theorem 18.1 (Singhof, 1993)**.** *Let k be a principal ideal domain. Then Theorem 0.3 holds additively.*

Corollary 18.2. *If additionally* $\operatorname{rk} G = \operatorname{rk} H + \operatorname{rk} K$, *then Theorem 0.3 holds multiplicatively as well.*

Proof. In this case one has $\operatorname{Tor} = \operatorname{Tor}^0 = H^*(BH) \otimes_{H^*(BG)} H^*(BK)$, so the ring map $H^*(BH) \otimes H^*(BK) \to \operatorname{Tor}$ is surjective. □

Singhof shows the total Pontrjagin p class of the tangent bundle of $K\backslash G/H$ lies in the image of Tor^0. Write $n = \operatorname{rk} G - \operatorname{rk}(H \times K)$ and $n' = \dim H\backslash G/K - n$. Through an inductive algebraic argument, Singhof shows $\operatorname{Tor}^{p,q}$ vanish outside the parallelogram with vertical edges $(p, q) \in 0 \times [0, n']$ and $-n \times [-2n, n' - 2n]$. In particular p_ℓ vanishes for $\ell > n'$. He further computes $\chi(H\backslash G/K)$, which can nowadays be more easily done using Kapovitch's model in Section 4.

19. Gugenheim–Munkholm

Gugenheim and Munkholm's joint work [**GMu74**] precedes Munkholm's solo effort and contains arguments that a homotopy-commutative diagram like (17.4) induces an additive isomorphism of Tors. It also states Munkholm's hypotheses for the diagram to homotopy-commute, at the end, but does not include a proof that they suffice, and the statement that Theorem 17.12 follows.

The main material, of which this suggested proof of Theorem 17.12 is an application, is focused on extending the definition and functoriality of Tor to a more general context than a span $M \leftarrow A \rightarrow N$ of DGA maps. The approach in Munkholm to this is that DGC maps $\mathbf{B}M \leftarrow \mathbf{B}A \rightarrow \mathbf{B}N$ are enough, for applying $\mathbf{\Omega}$ one gets $\mathbf{\Omega B}M \leftarrow \mathbf{\Omega B}A \rightarrow \mathbf{\Omega B}N$, defining $\mathrm{Tor}_{\mathbf{\Omega B}A}(\mathbf{\Omega B}M, \mathbf{\Omega B}N)$, and applying $\mathrm{Tor}_\varepsilon(\varepsilon, \varepsilon)$, this specializes to the $\mathrm{Tor}_A(M, N)$ one already had if one was lucky enough to have DGA maps $M \leftarrow A \rightarrow N$ to begin with.

Gugenheim–Munkholm, by contrast, assume M and N are respectively right and left DG A-modules and use the two-sided bar construction $\mathbf{B}(M, A, N)$ as a model for $\mathrm{Tor}_A(M, N)$. They generalize the notion of a map of DG modules, for M a right DG A-module and M' a right DG A-module, and $f \colon \mathbf{B}A \longrightarrow \mathbf{B}A'$ a DGC map, to what they call a ***f*-SH linear** map $M \Rightarrow M'$ This is a cochain map $g \colon \mathbf{B}(M, A, k) \longrightarrow \mathbf{B}(M', A', k)$ such that $g \otimes f$ makes the square with the comodule structure maps $\mathbf{B}(M, A, k) \rightarrow \mathbf{B}(M, A, k) \otimes \mathbf{B}A$ and $\mathbf{B}(M', A', k) \rightarrow \mathbf{B}(M', A', k) \otimes \mathbf{B}A'$ commute. There is a symmetric notion of f-SH linear map $h \colon N \rightarrow N'$ for a left DG A-module N and a left DG A'-module N', together inducing a chain map $g \otimes^f h \colon \mathbf{B}(M, A, N) \rightarrow \mathbf{B}(M', A', N')$ by the cotensor product and hence a map of Tors.

Gugenheim–Munkholm then introduce an appropriate notion of homotopy of f-SH linear maps, and show that given only homotopy-commutative squares as in (17.4) (λ_B standing in for f), one can actually *replace* the outer maps by maps which are λ_B-SH linear. Then the functorality of Tor from the previous paragraph, will give Munkholm's theorem 17.12, so long as the homotopies are known to exist.

20. The surjection operad

The remaining work requires further cochain-level operations generalizing the cup-i products. The value $c'(\sigma|_{\Delta^{[0,p]}})c''(\sigma|_{\Delta^{[p,p+q]}})$ of the cup product of homogeneous cochains $c', c'' \in C^*(X)$ and $c'' \in C^q(X; k)$ on a singular simplex $\sigma \colon \Delta^{p+q} \longrightarrow X$ can be seen as the value of $\mu_k(c' \otimes c'')$ on the diagonal $\sum_{j=0}^{p+q} \sigma|_{\Delta^{[0,j]}} \otimes \sigma|_{\Delta^{[j,p+q]}}$; the only term with the right dimensions not to evaluate to zero is $\sigma|_{\Delta^{[0,p]}} \otimes \sigma|_{\Delta^{[p,p+q]}}$. The higher Steenrod cup-i products are defined by apportioning the vertices differently; for example, to define $c' \smile_1 c''$ on $\sigma \colon \Delta^r \rightarrow X$, one sums, over all possible subdivisions $0 \leqslant p \leqslant q \leqslant r$, the product of the values of c' and c'' at the respective restrictions of σ to $\Delta^{[0,p] \cup [q,r]}$ and $\Delta^{[p,q]}$.[35] More generally, given n cochains $c^{(i)}$ and an ℓ-simplex, one can break up the vertex set $[0, \ell]$ into $m \geqslant n$ endpoint-overlapping intervals (in all possible ways) and assign some of the intervals to each $c^{(i)}$ (getting zero unless $|c^{(i)}| + 1$ is the number of vertices assigned). Multiplying over i and summing over subdivisions then yields a cochain. The resulting ***interval-cut operations*** $C^*(X)^{\otimes n} \longrightarrow C^*(X)$ are parameterized by the surjections $\varphi \colon \{1, \ldots, m\} \longrightarrow \{1, \ldots, n\}$ assigning subintervals to cochains.

Example 20.1. Identifying φ with the sequence $\big(\varphi(1), \ldots, \varphi(m)\big)$, the cup product corresponds up to sign with the sequence $(1, 2)$, the cup-1 product to $(1, 2, 1)$, and the cup-i product in general to an alternating sequence $(1, 2, 1, 2, \ldots)$ of length $i+2$.

One considers only sequences with no two consecutive entries equal, to avoid producing degenerate simplices, and redefines the notion of cochain accordingly. The ***normalized cochain algebra*** $C^*(X_\bullet; k)$ on a simplicial set $X_\bullet$ is the DG subalgebra containing all and only cochains vanishing on each degenerate simplex.

[35] We ignore a sign here.

It has been shown that the interval-cut operations on $C^*(X_\bullet;k)$ are closed under the action of the symmetric group and composition, and that the differential of such an operation is a linear combination of other such operations [**BF04**, Prop. 1.2.7][**McS03**, Props. 2.18, 19, 26].

In the language of operads[36] they form a symmetric DG-operad $\mathscr{X}$, called the ***surjection operad*** [**BF04**]. The normalized cochain algebra of a pointed simplicial set is then functorially a $\mathscr{X}$-algebra [**McS03**, Thm. 2.15].

Definition 20.2 ([**Fr20a**, §3.2]; [**Kad03**, §2.1] over $\mathbb{F}_2$; [**Fr20b**, (3.13)] for signs)**.**

A ***homotopy Gerstenhaber algebra*** (HGA) is an algebra over the symmetric DG operad $F_2\mathscr{X}$ of $\mathscr{X}$ generated by the DGA product $(1,2)$ and the operations E_ℓ corresponding to $(1,2,1,3,1,\ldots,1,\ell+1,1)$.[37] An ***extended HGA*** is an algebra over the symmetric DG operad $F_3'\mathscr{X}$ of $\mathscr{X}$ generated by $F_2\mathscr{X}$ and the operations $F_{p,q}$ corresponding to

$$(1,\ p+1,\ 1,\ p+2,\ 1,\ p+3,\ \ldots,\ 1,\ p+q,\ 1,\ p+q,\ 2,\ p+q,\ 3,\ p+q,\ \ldots,\ p,\ p+q).$$

An (***extended***) HGA ***homomorphism*** $f\colon A\longrightarrow B$ is a DGA map distributing over the operations ($F_{p,q}$ and) E_ℓ.

To explain the notation, the DG suboperad $F_2\mathscr{X}$ is a term of a certain increasing filtration $F_n\mathscr{X}$ on $\mathscr{X}$ with $F_3'\mathscr{X}\subseteq F_3\mathscr{X}$.[38,39]

Example 20.3. A CDGA A is canonically an $\mathscr{X}$-algebra with $F_\ell\mathscr{X}\backslash F_{\ell-1}\mathscr{X}$ acting identically as 0 for $\ell\geqslant 2$. In particular, a CDGA A is naturally an extended HGA.

Corollary 20.4. *For any pointed simplicial set $X_\bullet$, its algebra $C^*(X_\bullet)$ of normalized cochains is also naturally an extended* HGA *with $-E_1=\smile_1$ and $-F_{1,1}=\smile_2$.*

An HGA structure on a DGA A is known to induce a multiplication $\mu_{\mathbf{B}A}$ on the DGC $\mathbf{B}A$ rendering it a DG Hopf algebra.[40]

Theorem 20.5 (Franz [**Fr20a**])**.** *A homotopy Gerstenhaber algebra A admits a natural* DGC *map $\Phi_A\colon\mathbf{B}(A\otimes A)\longrightarrow\mathbf{B}A$ satisfying the unitality and associativity*

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[36] Here, we use the terminology as a kind of shorthand, and as a familiar reference point for those familiar with operads, but for the reader who has not, it is enough to understand "a system of composable operations (of varying arities) closed under composition." These satisfy a long but intuitive set of axioms and give a formal organizing notion for certain algebraic structures.

[37] HGAs were first defined [**GeV95**] as DGAs equipped with operations E_ℓ satisfying certain axioms. It is a theorem [**McS03**, Thm. 4.1][**BF04**, §1.6.6] that this yields precisely $F_2\mathscr{X}$-algebras.

[38] $F_0\mathscr{X}$ contains functions with 0 or 1 value. $F_\ell\mathscr{X}\backslash F_{\ell-1}\mathscr{X}$ contains sequences such that ℓ is the maximum number of alternations in a two-value subsequence. For example, $(1,2)$ lies in $F_1\mathscr{X}$ and $(3,1,4,3,4,2,1,2)$ in $F_3\mathscr{X}$, with maximum alternation attained by the subsequence $(3,1,3,1)$.

[39] For those who know what this is, $F_n\mathscr{X}$ is equivalent to the cochains on the operad of little n-cubes. In fact [**BF04**, Thm. 1.3.2, Lem. 1.6.1], $\mathscr{X}$ is a quotient of the DG-operad $\mathscr{E}$ associated to the classical Barratt–Eccles simplicial operad, which is filtered by a sequence $F_n\mathscr{E}$ of E_n-operads (of which the $F_n\mathscr{X}$ are the images), so the normalized cochain algebra is an E_∞-algebra.

The filtrands $F_n\mathscr{X}$ had been already identified and shown to be equivalent to the little n-cubes operads by other methods [**McS03**] before this surjection was found. Earlier still, McClure–Smith had shown $F_2\mathscr{X}$ is equivalent to the little squares operad [**McS02**] in order to prove Deligne's conjecture that the Hochschild cohomology of a ring is naturally an E_2-algebra.

[40] Extended HGAs over $\mathbb{F}_2$ were first studied by Kadeishvili [**Kad03**], seeking conditions on an HGA A under which $\mathbf{B}A$ admitted cup-i products. He gave a characterization in terms of operations of $\mathscr{X}$ acting on A; for $i=1$, these are the $F_{p,q}$.

axioms for SHC*-algebras and such that* $\mu_{\mathbf{B}A} = \Phi_A \circ \nabla$*. If* A *is an extended homotopy Gerstenhaber algebra, then* Φ_A *is an* SHC*-algebra structure.*

This SHC-algebra structure is not complicated to define, but the fact it actually satisfies the axioms is an extraordinarily complex computational result (found by hand but verified by computer) for which we still have no conceptual explanation.

21. Franz

In 2019, Franz [**Fr21a**] proved the following.

Theorem 21.1 (Franz)**.** *Let* k *be principal ideal domain in which* 2 *is a unit. Then Theorem 0.2 holds.*

Recall that the characteristic-0 multiplicative results work because there exist DGA structures on the complexes computing Tor. Kadeishvili–Saneblidze [**KadS05**, Thm. 7.1] found a DGA structure on the twisted tensor product $\mathbf{B}A \otimes_t A'$ when A and A' are HGAs and $t = t^{A'} \circ \mathbf{B}\phi$ for an HGA homomorphism $\phi\colon A \to A'$. This DGA structure is functorial on the category of HGA diagrams of shape $\bullet \to \bullet$. By Theorem 5.2, Definition 11.5, and Corollary 20.4, $\mathbf{B}C^*(BG) \otimes_{t^{C^*(BK)}\mathbf{B}\rho} C^*(BK)$ is a DGA with cohomology ring $H^*(G/K)$, and $\mathbf{B}H^*(BG) \otimes_{t^{H^*(BK)}\mathbf{B}H^*\rho} H^*(BK)$ is a DGA with cohomology ring $\operatorname{Tor}_{H^*BG}(k, H^*BK)$. Using the Halperin–Stasheff maps (12.2), Franz considers the composite cochain map

$$\begin{array}{ccc} \Theta\colon \mathbf{B}H^*(BG) \underset{t^{H^*(BK)}\,\mathbf{B}H^*\rho}{\otimes} H^*(BK) & \xrightarrow{\mathrm{id}\otimes\lambda_H^{(1)}} & \mathbf{B}H^*(BG) \underset{t^{C^*(BK)}\,\lambda_H\,\mathbf{B}H^*\rho}{\otimes} C^*(BK) \\ & & \Big\downarrow \wr\ \delta_h \\ \mathbf{B}C^*(BG) \underset{t^{C^*(BK)}\,\mathbf{B}\rho}{\otimes} C^*(BK) & \xleftarrow[\lambda_G\otimes\mathrm{id}]{} & \mathbf{B}H^*(BG) \underset{t^{C^*(BK)}\,\mathbf{B}\rho\,\lambda_G}{\otimes} C^*(BK) \end{array}$$

where $\lambda_H^{(1)} = t^{H^*(BK)} \circ \lambda_H \circ s^{-1}_{C^*(BK)}$ and δ_h is the cap product with the homotopy h of twisting cochains $\mathbf{B}C^*(BG) \longrightarrow H^*(BK)$ implied by the commutativity of (12.3), as proved by Munkholm. By a variant of Proposition 5.4, $H^*(\Theta)$ is a linear isomorphism.

Because λ_G and λ_H are not multiplicative, Θ may not preserve the DGA structures, but we follow Wolf in postcomposing a formality map to kill $\smile_1$-products preventing λ_G from being multiplicative. Let T be a maximal torus of H and set $\phi = B(T \hookrightarrow H)$. Franz follows Θ with

$$\Psi\colon \mathbf{B}C^*(BG) \underset{t^{C^*(BK)}\,\mathbf{B}\rho^*}{\otimes} C^*(BK) \xrightarrow{\mathrm{id}\otimes f\phi^*} \mathbf{B}C^*(BG) \underset{t^{C^*(BT)}\,\mathbf{B}(f\phi^*\rho^*)}{\otimes} H^*(BT),$$

where $\mathrm{id} \otimes \phi^*$ induces an injection in cohomology and $\mathrm{id} \otimes f$ is a quasi-isomorphism by Proposition 5.4. Since ϕ is an HGA map, if f can also be chosen to be an HGA map, Ψ will be multiplicative by functoriality of the DGA structure.

The existing Gugenheim–May map does not meet these desiderata, so Franz defines a certain ideal $\mathfrak{k}_X$ of undesirable cochains, functorial in spaces X, and then constructs a new formality map $f\colon C^*(BT) \longrightarrow H^*(BT)$ which is an HGA map annhilating $\mathfrak{k}_{BT}$ *so long as* 2 *is a unit of* k. At the same time, using the fact $C^*(BT)$ is an extended HGA, he applies Theorem 20.5 to obtain an SHC-algebra structure Φ such that the DGC quasi-isomorphism $\lambda_T\colon \mathbf{B}H^*(BT) \longrightarrow \mathbf{B}C^*(BT)$

compiled with respect to Φ has image in the error ideal $\mathfrak{k}_{BT}$. Moreover, the compiled maps make (12.3) commute up to a homotopy h such that $t^{C^*(BK)}h$ is congruent to the identity modulo $\mathfrak{k}_{BK}$. Further, λ_G is an SHC-algebra map in the sense of making (17.1) homotopy-commute via a DGC homotopy whose associated twisting cochain homotopy takes the coaugmentation coideal into $\mathfrak{k}_{BG}$. This twisting cochain homotopy is hence annihilated by postcomposing with $f\phi^*\rho^*$. With these choices of SHC-algebra structure and formality map, one checks postcomposing with Ψ simplifies Θ so that the composite $\Psi\Theta$ is

$$\lambda_G \otimes \phi^* \colon \mathbf{B}H^*(BG) \underset{H^*(\rho)t^{H^*(BG)}}{\otimes} H^*(BK) \longrightarrow \mathbf{B}C^*(BG) \underset{f\phi^*\rho^* t^{C^*(BG)}}{\otimes} H^*(BT).$$

Since $H^*(\Psi)$ is a multiplicative injection and $H^*(\Theta)$ is a bijection, to show $H^*(\Theta)$ is multiplicative, it suffices to show $H^*(\Psi\Theta)$ is. When A' is a CDGA, the multiplication on the twisted tensor product $\mathbf{B}A \otimes_t A'$ making it a DGA is just the naive multiplication permuting the tensor factors and then multiplying components using $\mu_{\mathbf{B}A} \otimes \mu_{A'}$, because the commutativity of A' means the higher-order HGA operations on A' figuring in the multiplication formula vanish. To show $H^*(\Psi\Theta)$ is multiplicative, it is thus enough to construct a DGC homotopy between the two paths around the large diagram

$$\begin{array}{ccccc}
(\mathbf{B}H^*BG)^{\otimes 2} \otimes (H^*BK)^{\otimes 2} & \xrightarrow{\nabla \otimes \mathrm{id}} & \mathbf{B}(H^*BG)^{\otimes 2} \otimes (H^*BK)^{\otimes 2} & \xrightarrow{\Phi \otimes \mu} & \mathbf{B}H^*BG \otimes H^*BK \\
\downarrow {\scriptstyle \lambda_G^{\otimes 2} \otimes (\phi^*)^{\otimes 2}} & & \downarrow {\scriptstyle \lambda_G \otimes \lambda_G \otimes (\phi^*)^{\otimes 2}} & & \downarrow {\scriptstyle \lambda_G \otimes \phi^*} \\
(\mathbf{B}C^*BG)^{\otimes 2} \otimes (H^*BT)^{\otimes 2} & \xrightarrow[\nabla \otimes \mathrm{id}]{} & \mathbf{B}(C^*BG)^{\otimes 2} \otimes (H^*BT)^{\otimes 2} & \xrightarrow[\Phi \otimes \mu]{} & \mathbf{B}C^*BG \otimes H^*BT.
\end{array}$$

By Theorem 20.5, the product on $\mathbf{B}C^*(BG)$ factors as $\Phi \circ \nabla$, so it is enough to find a DGC homotopy for the right square. The DGA coordinates commute on the nose since ϕ^* is a ring map. For the bar coordinates, λ_G is an SHC-algebra map via a homotopy H such that the coaugmentation coideal of $\mathbf{B}(H^*BG)^{\otimes 2}$ is annihilated by the twisting cochain $t^{H^*(BT)}f\phi^*\rho^*H$ defining the twisted differential on the codomain. An easy lemma then shows $H \otimes \phi^*$ is the desired DGC homotopy.

Example 21.2. Let $H \cong \mathrm{U}(1)$ be the subgroup of $\mathrm{SU}(4)$ with diagonal entries $\mathrm{diag}(z^{-3}, z, z, z)$. Then, indexing generators by degree, we have a ring isomorphism

$$H^*\big(\mathrm{SU}(4)/H; \mathbb{Z}[\tfrac{1}{2}]\big) \;\cong\; \frac{\mathbb{Z}[\frac{1}{2}][s_2] \otimes \Lambda[a_5, b_7]}{(3s^2, s^3, s^2a_5)}.$$

22. Carlson–Franz

On seeing Franz's paper, the present author at once had the insight that Franz's proof of Theorem 0.2 would effortlessly generalize to a proof of Theorem 0.3. That insight was wrong: Franz pointed out no one had defined a multiplication μ on a two-sided twisted tensor product $A'' \otimes_{t''} \mathbf{B}A \otimes_{t'} A'$ generalizing the product of Kadeishvili–Saneblidze. One also needs a version of the Eilenberg–Moore theorem preserving μ; more precisely, one needs to show that in the situation of Theorem 5.2, the relevant instance $C^*(X) \otimes_{t''} \mathbf{B}C^*(B) \otimes_{t'} C^*(E) \to C^*(X) \otimes C^*(E) \to C^*(Y)$ of the map of (5.1) is multiplicative up to a homotopy h.

Within a month, however, Franz shared formulas for μ and h he believed should work once a correct choice of signs was discovered. The present author guessed the signs and proved that these formulas work as expected.[41] After that, the proof summarized in the preceding section works *mutatis mutandis*. This was the work presented at the conference session.

Theorem 22.1 ([**CaF21**]). *Theorem 0.3 holds whenever* 2 *is a unit of* k.

As one needs to invert 2 for this approach to work, this result is not strictly an improvement on Munkholm's additive theorem 17.12 in all cases.

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