Return of the Integral People

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All of these were problems for MAT236 Vector Calculus at the University of Toronto at Mississauga at some point during the semesters I taught it.

1. You are sitting quietly on an intercity bus staring lovingly into your phone, happy to live in an age where public interaction with strangers is almost entirely avoidable, when despite there being many other available seats on the bus, someone sits next to you. Briefly confused, you give no outward notice of the intrusion and reabsorb yourself in Facebook, the calm yet strong blue of the navigation bar comforting you, *rooting* you. The rest of the world fades away. The tread of the wheels over the road grit dies down to a low murmur. "Haha," you click. "Like." Soon you forget your peril.

   But the intruder insists on communication, and speaks directly into your ear. It appears they are a Length Person:

   > When secured at its two ends and otherwise hanging freely under its own weight, a chain hangs in the shape of a *catenary curve*, the graph of the function $f(x) = \frac{c}{2}(e^{x/c} + e^{-x/c})$ for some positive real number c. Fix a positive real number a and consider a chain hanging between the points $\big(0, f(0)\big)$ and $\big(a, f(a)\big)$. What is its length?

   You know from experience, that cruelest of teachers, that were you to move to another seat, this person would only follow you. Answer their question so that you can forget them and return to something of real, lasting value.

2. You are strolling down Bloor Street on a pleasant summer day, minding your own damned business. On the sidewalk in the distance, you see a dishevelled-looking man sitting on a blanket, with a dog, a bowl of water, and a cardboard sign you can't quite make out from so far away. Your plans unavoidably take you past this point, which it is far too difficult to reasonably circumnavigate; and despite your awareness that this gentleman suffers from systemic societal problems far too severe to be more than temporarily meliorated by any individual act of charity, and your conviction that the city of Toronto owes its citizens better social services (along with a nagging suspicion that capitalism inevitably creates such suffering, and so must be torn down and replaced with something better—but what?), you from long experience know full well how this situation will play out: your conscience is about to make you slightly poorer.

   In the event, however, it transpires you have gravely misjudged. As you near this down-on-his-luck gentleman and gain a better vantage on his comportment and sign, you realize with a sickening start that he is not interested in your money at all. The situation is far worse than you had dared anticipate: this man's problems are not societal, but mathematical—for he is an Integral Person! But as so often in this life, it is now too late for you to avoid him and his pathetic yet insistent plea:

> Compute the integral $\int_\sigma yz\,dx + xy\,dy + xz\,dz$, where the parameterized curve σ is defined by $\sigma(t) = (\csc t, \sin t, \cot t)$ for $t \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

Other passersby shuffle past with a mixture of condescension and alarm. The Integral Man has been out here all day, warbling this plaint unheeded at the feet of callous strangers. Help him, as only you can.

3. At a party, you are cornered by a Complex Integral Person, whose difficulties are still greater than those of the garden-variety Integral Person one meets in day-to-day life. It speaks:

   > Suppose a complex-valued function $f(z) = u(x,y) + iv(x,y)$ is defined and holomorphic on an open region R in the complex plane containing a piecewise smooth closed curve C and the region D it bounds. Using Green's theorem and the definition of a holomorphic function, prove Cauchy's theorem that $\int_C f = 0$.

   There is no polite way to leave this conversation, and in any event your agility and coordination are not what they were when you arrived. You have no choice but to, recalling notation and concepts from the first two problems of the earlier assignments, satisfy this person's demand.

4. You are alone in a dark alley in a bad neighborhood. An Integral Baby crawls up alongside you, and with its underdeveloped little mouth tries to form the following question:

   > Consider the curve $c(t) = (t+2, 3t-2)$, parameterized over $t \in [1,2]$. Can you please help me compute the scalar integral $\int_c (x+y)\,ds$?

   Spent from the effort of communicating this important request, the baby then breaks down into a jag of sobbing and infantile gibberish. You could actually easily escape this time; this baby's uncoordinated movements and chubby little limbs could not possibly take it anywhere fast. In a race, you would absolutely *own* this baby. But then, would that be fair? What kind of person are you, anyway? Have a heart and answer its tiny little question.

5. You awake in the middle of the night to feel yourself pinned to the bed, the sensation of hot, rank breath on your cheek. Your light switches on. With a jolt, you open your eyes to see a creature looming over you, its nose almost touching yours, its sweat dripping onto you. It gazes at you, its bloodshot eyes filled with a indescribable, terrible *need*. You're annoyed, of course: this happens far too often, and it will only let you sleep again once its integral is computed. It hisses:

> Let $C \subsetneq \mathbb{R}^3$ be the curve parameterized by $\gamma(t) = \left(te^{e^{5t}}, t - \frac{\pi}{2}, \cos^5(t/2)\right)$ for $t \in [0, \pi]$ and compute the line integral $\displaystyle\int_C 2xz \sin y \, e^{x^2 z} \, dx + e^{x^2 z} \cos y \, dy + x^2 e^{x^2 z} \sin y \, dz$.

With a sigh you reach for the notebook you keep by your bedside for such occasions and begin to calculate . . . .

6. An Area Person appears before you, shaking geometrically. Clutching the mysterious pendant they are wearing, they say it can be described by the inequality $x^2 + y^2 + z^2 \leqslant R^2$ for some real number $R > 0$ and they desperately need to determine its surface area. Please help them (perhaps remembering polar coordinates).

7. A Volume Person appears after this person, gesticulating wildly.

   > Consider the octahedron in $\mathbb{R}^3$ whose six vertices are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$. Please compute its volume as a double integral. (Hint: First find the volume of the subregion where $x, y, z \geqslant 0$.)

8. An old friend texts you that they want to talk on the phone sometime. You've known them long enough that even through the tone-deaf medium of SMS, you can read between the lines: though they're trying to sound chill about it, something has deeply upset them. It seems urgent, so you leave class to call. What's gone wrong? How can you help? After some forced banter, plainly peripheral to the main thrust, they come clean with their concern.

   > "I can't sleep anymore. The differential 1-form
   >
   > $$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}$$
   >
   > is equal to the exterior derivative
   >
   > $$d\theta = d\left(\arctan \frac{y}{x}\right),$$
   >
   > so the integral of ω around the unit circle C should be zero. But it's not! Is everything I know a lie?!"

   Your mind races. You'd had no idea your friend was an Integral Person (not that there's anything wrong with that, but how could you go so long without learning something so fundamental? How long have *they* known? You have a million questions, but you know to not ask right now, just be accepting and supportive.)—but then, this just underscores the utility of Vector Calculus in everyday life.

   (a) Verify (again; if you've done this problem previously, this should be very fast anyway) your friend's statement about the external derivative.

   (b) Compute the integral they claim is nonzero.

   (c) Explain why your friend might think this contradicts the results we've learned.

(d) Placate your friend by explaining why there is no contradiction.

(e) Explain why the observation that $\arctan(y/x) = \theta$, where defined, might have enabled us to compute $\int_C \omega$ more easily. (Hint: the form ω *does* admit a well-defined, smooth antiderivative on the region $\varepsilon < \theta < 2\pi - \varepsilon$ for small positive ε; what happens if we compute the integral only over a portion of the unit circle in that region?)