Closed images of proper maps

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A map $f\colon X \to Y$ of topological spaces is said to be *proper* if the inverse image $f^{-1}(K)$ of every compact subset K of the target space Y is also compact. It is claimed by Bott and Tu that the image of a continuous, proper map must be closed.[1] This turns out to be true under reasonable hypotheses, though in full generality, somewhat artificial counterexamples can be constructed. We will call a continuous, proper map *bad* if its image is not closed. After giving some examples of bad maps, we discuss hypotheses under which they cannot occur.

We make a general observation about bad inclusions. Note that inclusions are not in general proper maps; $(0,1) \hookrightarrow [0,1]$ is one simple counterexample.

We give thanks to George Leger for helpful conversations about this note.

Lemma. *Suppose $f\colon X \to Y$ is a bad map with image F. Then the inclusion $F \hookrightarrow Y$ is bad. Conversely, given a bad inclusion $F \hookrightarrow Y$, composing with any continuous, proper surjection $X \twoheadrightarrow F$ yields a bad map $X \to Y$.*

Proof. For each compact $K \subset Y$, the intersection $K \cap F = f\big(f^{-1}(K)\big)$ is compact, so the inclusion $F \hookrightarrow Y$ is another proper map. Since F is not closed in Y, the inclusion is bad.

For the converse, note that the composition of two proper maps is again proper. □

Call a subset $F \subset Y$ a *bad subset* if the inclusion $F \hookrightarrow Y$ is bad; the lemma shows the study of bad maps reduces to that of bad subsets. Now if a space is not Hausdorff, a compact subset need not be closed, so even a compact subset can be bad. We present the simplest example.

Example 1. *There exists a space Y admitting a bad map from every compact space X.*

Proof. Let Y be the *Sierpiński space* $\{a, b\}$, with open sets $\varnothing$, $\{a\}$, and $\{a, b\}$. The singleton $\{a\}$ is not closed, but every subset of Y meets $\{a\}$ in a finite and hence compact set, so $\{a\}$ is bad. Now any constant map from a compact space X to a is proper. □

In fact, an inclusion of a compact set into a non-Hausdorff space need not even be proper.

Example 2. *There exists a space Y with two compact subsets K_1, K_2 such that $K_1 \cap K_2$ is not compact.*[2]

Proof. Topologize $Y = [0,1]$ by taking the open sets to be all subsets of $(0,1)$ and all cofinite sets containing at least one of 0 and 1. Then $[0,1)$ and $(0,1]$ are compact, but their intersection $(0,1)$ is not. □

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[1] [1, p. 41, Exercise 4.10.1]

[2] [2, p. 283]

The situation changes when the target space is Hausdorff.

Proposition. *If Y is Hausdorff, then the inclusion $F \hookrightarrow Y$ of a compact set is proper with closed image.*

Proof. Since a compact subset of a Hausdorff space is closed, F is closed. Any other compact subset K is also closed, so $K \cap F$ is a closed subset of a compact set, and hence compact. □

Thus a bad subset of a Hausdorff space cannot be compact.

Proposition. *Let X be a topological space, Y a first-countable Hausdorff space, and $f\colon X \to Y$ a continuous, proper map. Then the image F of f is closed in Y.*

Proof. Suppose $F \hookrightarrow Y$ is a proper map and y is a limit point of F; we show $y \in F$. Since Y is first-countable, y has a countable neighborhood base of nested open sets $U_n \supset U_{n+1}$ ($n \in \mathbb{N}$), and there is for each n a point $y_n \in (U_n \cap F) \setminus \{y\}$. Write S for the subset $\{y_n \mid n \in \mathbb{N}\}$ of F. The set $K := S \cup \{y\}$ is compact, for any open set containing y contains a U_n and hence all y_k for $k \geq n$; but because S fails to contain its limit point y, it is not closed, and because Y is Hausdorff, S is not compact. But by properness, $K \cap F$ is compact, so it follows that $y \in F$. □

Proposition. *Let X be a topological space, Y a locally compact Hausdorff space, and $f\colon X \to Y$ a continuous, proper map. Then the image F of f is closed in Y.*

Proof. Suppose $F \hookrightarrow Y$ is a proper map and y is a limit point of F; we show $y \in F$. Since Y is locally compact, there is neighborhood U of y contained in a compact K. By properness, $K \cap F$ is compact. Because Y is Hausdorff, $K \cap F$ is closed, and $U \cap F$ relatively closed in U. But then $U \cap F$ contains its limit point y.[3] □

Without such hypotheses on the target space, the image of a continuous, proper map need not be closed.

Example 3. *There exists a continuous, proper map from a discrete space to a T_5 space, with image not closed.*

Proof. Let $Y = X \amalg \{p\}$ be an uncountable set. Make Y into a *fortissimo space*[4] by declaring all subsets of X to be open and the open neighborhoods of p to be all cocountable sets U_p, U'_p, etc. containing p. Any set containing a U_p is cocountable, an intersection $U_p \cap U'_p$ is cocountable, and the intersection of a U_p with a subset of X is another subset of X, so the definition works. Another way of putting it is that

$$C \subset Y \text{ is closed} \iff (p \in C \quad \text{or} \quad C \text{ is a countable subset of } X). \tag{1}$$

To show Y is T_5, we need that it is T_1 and completely normal. Y is T_1 by (1), because $p \in \{p\}$ and singletons are countable. To show Y is completely normal, let A and B be a *separated* pair of sets, meaning that $A \cap \overline{B} = \overline{A} \cap B = \varnothing$; we must show there are disjoint open sets $U \supset A$ and $V \supset B$. This breaks down by cases:

- If A and B are in X, both are open by definition.
- If p is in one of the sets, say A, then since $A \cap \overline{B} = \varnothing$, we have $p \notin \overline{B}$. Then $\overline{B} \subset X$, so $\overline{B}$ is open. And since $\overline{B}$ is closed, $Y \setminus \overline{B} \supset A$ is open.

[3] To see y is a limit point of $U \cap F$ in U, note that any neighborhood of y in U is of the form $U \cap V$ for another neighborhood V of y, and that since y is a limit point of F in Y, we know $U \cap V \cap F \neq \varnothing$.

[4] The name is a pun on *Fort spaces*, named after Marion Kirkland Fort, Jr.; see [3, pp. 53–54]. These come in two major variants, depending on whether X is countable or uncountable. The open sets are defined in the same way except that the neighborhoods of p are instead required to be cofinite. Fort spaces turn out to be compact, and the countable ones metrizable [3, pp. 52–53].

A compact subset K of Y must be finite: if not, K contains a countably infinite $C \subset X$, which is closed by (1), and hence compact. X is discrete in the subspace topology, but the only compact subsets of a discrete space are finite, contradicting our assumption C was infinite. Conversely, any finite set is compact. Thus $K \cap X$ is compact for each compact $K \subset Y$, so the inclusion $X \hookrightarrow Y$ is proper. But X is uncountable and does not contain p, so by (1), it is not closed. □

Of course, a fortissimo space is neither locally compact nor first-countable.

References

[1] Raoul Bott and Loring W. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics **82**. Springer-Verlag, New York, 1982.

[2] Norman Levine, "On the intersection of two compact sets." *Rend. Circ. Mat. Palermo* **17** (1968), no. 3, 283–288. Available online at <link.springer.com/article/10.1007%2FBF02909627>.

[3] Lynn Arthur Steen and J. Arthur Seebach, Jr., *Counterexamples in Topology*. Springer-Verlag, New York, 1978. Reprinted by Dover Publications, New York, 1995.