Research statement

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My work lies within the confines and the interface of topology, geometry, and algebra, from symplectic geometry and complex bordism on the one end to A_∞-algebras and Galois cohomology on the other. A unifying theme is continuous group actions on smooth manifolds.

My first mature work applied an invariant called *Borel cohomology* to situations of long-standing geometric interest, for example the *isotropy actions* on *(generalized) homogeneous spaces*, the *biquotients* of two-sided actions, and *cohomogeneity-one actions*. Highly symmetric spaces like these tend to have interesting geometry, and examples of Riemannian manifolds with non-negative curvature tend to be symmetric in some way or another. Aside from their inherent interest, actions, especially by tori, also provide a tool to simplify computations of integrals over manifolds and their algebro-topological invariants.

Trying to answer a question about circular symmetries of homogeneous spaces led me to accidentally reprove a poorly publicized classical result on the rational cohomology of homogeneous spaces and then (intentionally) write a textbook on the subject more generally, currently under revision for Springer, as discussed in Section 1. (For some projects in rational homotopy theory these questions have led to, see Section 7.2 and Section 7.3.)

Equivariant K-theory gives a more refined but less tractable invariant of actions; see Section 3. A fair deal of my work involves an important family of conditions on the Borel cohomology and equivariant K-theory of an action called *equivariant formality*, encountered in common situations such as that of a Hamiltonian torus action on a compact symplectic manifold, and discussed in Section 4. Some of this work has had geometric corollaries for vector bundles over spaces with a cohomogeneity-one action [AmGÁZ19, Thm. A(1)]. I have also proved that in the case of isotropy actions, the notion of equivariant formality is linked with a general notion of *formality* also implicated in the A_∞-algebraic work discussed in Section 2.

To extend my thesis results to coefficient rings other than $\mathbb{Q}$, a less rigid notion of formality is called for, leading to the study of *A_∞-algebras* and other up-to-homotopy algebra structures. Learning this material led me, surprisingly, to new results on Tor and the bar construction resolving questions left unanswered since the 1970s. My results along these lines so far in this field are summarized in Section 2. My work with A_∞-algebras and formality is expected to aid in some computations in *Galois cohomology*; see Section 7.1 for a prospectus.

Equivariant topology in general depends heavily on fixed point sets of group actions, and one can ask which of "local models" of potential actions near fixed point sets actually arise as fixed point data of actions on manifolds. Such questions lead into the realm of *equivariant cobordism*, which attempts to understand not symmetries of individual spaces, but the totality of *all* spaces admitting certain kinds of symmetries. My collaborators and I have been particularly successful with a commonly encountered and widely studied class of well-behaved actions called *GKM actions*. My work and plans in this area are described in Section 5. Among the latter, I would like to explicitly describe in terms of generators and relations the coefficient rings for *torus-equivariant complex cobordism*, whose central role and many of whose properties are known, but whose structure is only partially understood.

I have also studied the non-equivariant topology of certain spaces of interest to symplectic geometers, the fibers of *Gelfand–Zeitlin systems*, and this has led to a sort of weak local model for a coadjoint orbit of a unitary or special orthogonal group, which is expected to lead to a solution to a long-standing open question regarding geometric quantization. See Section 6 for a discussion of this work.

Early forays into low-dimensional topology and dynamical systems [Ca10, AkC12] will not be discussed here.

1. Rational cohomology of homogeneous spaces

A ***homogeneous space*** is the orbit of a single point under a Lie group action. The geometry of such a space is highly symmetric, being identical at every point, and homogeneous spaces have long been studied by differential geometers. The most famous algebraic result about homogeneous spaces may be H. Cartan's result that if G and a subgroup K are compact, connected Lie groups, then the real cohomology ring of the homogeneous space G/K is given by

$$H^*(G/K;\mathbb{R}) \cong \mathrm{Tor}^*_{H^*(BG;\mathbb{R})}\big(\mathbb{R}, H^*(BK;\mathbb{R})\big).$$

In order to characterize equivariant formality of isotropic circle actions (see Section 4.3) in my thesis, I derived a consequence of Cartan's theorem which states that unless a circle subgroup S^1 of a connected Lie group G is nullhomotopic in G, the rational cohomology ring of G/S^1 is isomorphic to that of a product $S^2 \times \prod S^{2n_\ell+1}$ of spheres [Ca19a, Appendix A].

1.1. My book

My later discovery that this result is not actually original, having been announced without proof by Leray and Koszul in the late 1940s, solidified what had been growing into a general discontent with the secondary literature in this area. Convinced that this material needed to be better publicized, I resolved to write a text on the rational non-equivariant cohomology of homogeneous spaces [Ca15], which I have since submitted for publication with Springer. I am now preparing a revision with more background material on Lie groups at the request of the editors.

The manuscript uses a touch of rational homotopy theory to streamline the approach to the "Cartan" cochain model for $H^*(G/K;\mathbb{Q})$ which Borel developed in his thesis, and is meant in part to be a gentle introduction to spectral sequences suitable for a second-year graduate student. The necessary algebra is developed along the way and the resulting exposition is substantially faster than previously published accounts. Several aspects of my approach do not seem to appear elsewhere.

2. A_∞-algebraic methods

Many of my theorems about (equivariant) cohomology of various spaces depend on rational or real coefficients, which are necessary because there are no functorial commutative differential graded algebras computing cohomology with $\mathbb{Z}$ or $\mathbb{F}_p$ coefficients. For instance, the rational analogue of Cartan's ring isomorphism $H^*(G/K;\mathbb{R}) \cong \mathrm{Tor}^*_{H^*(BG;\mathbb{R})}\big(\mathbb{R}, H^*(BK;\mathbb{R})\big)$, a centerpiece of the monograph mentioned in Section 1.1, is proved using such models. Eilenberg and Moore had proven a ring isomorphism $H^*(G/K;k) \cong \mathrm{Tor}^*_{C^*(BG;k)}\big(k, C^*(BK;k)\big)$, and that the right-hand side is the target of a spectral sequence starting at $E_2 = \mathrm{Tor}^*_{H^*(BG;k)}\big(k, H^*(BK;k)\big)$. In this light, Cartan's isomorphism says that for $k = \mathbb{R}$ or $\mathbb{Q}$, this spectral sequence collapses with no multiplicative extension problem. The collapse is proven using commutative models of $C^*(BG;\mathbb{R})$ and $C^*(BK;\mathbb{R})$ and explicit differential graded algebras $H^*(BG;\mathbb{R}) \longrightarrow C^*(BG;\mathbb{R})$ and $H^*(BK;\mathbb{R}) \longrightarrow C^*(BK;\mathbb{R})$ inducing isomorphisms in cohomology to induce a map of Tors.

There are no such maps when we replace $\mathbb{R}$ with an arbitrary principal ideal domain k. The key workaround is to (1) reduce the requirement of commutativity to some weaker, up-to-homotopy notion, (2) find some sort of up-to-homotopy homomorphisms $H^*(BG;k) \longrightarrow C^*(BG;k)$ and $H^*(BK;k) \longrightarrow C^*(BK;k)$, and then (3) extend the functoriality of Tor to encompass this sort of weak map so that an isomorphism of Tors is still induced. Sugawara initiated the study of these notions in the late 1950s, which was substantially extended and reformulated in work of Clark, Gugenheim, Munkholm, Halperin, and especially Stasheff in the 1960s and '70s. Various completions of the program of extending Cartan's isomorphism were ultimately completed independently by Munkholm, Gugenheim–May, Husemoller–Moore–Stasheff, and Wolf, all around 1974. All had the weakness that the isomorphism was only additive, yielding only the graded k-module structure on $H^*(G/K;k)$.

My work in 2020–2021 re-examined these notions to prove multiplicativity. A key contribution was to define a product on Tor in more general circumstances and show that with this product and the existing notions of homotopy-commutative maps, maps of Tor including the known isomorphism become multiplicative.

Theorem 2.1 ([Ca22a]). *Let $A' \leftarrow A \rightarrow A''$ be maps of strongly homotopy-commutative k-algebras.*[1] *Then the graded k-module $\operatorname{Tor}_A(A', A'')$ carries a ring structure functorial in triples of maps of strongly homotopy-commutative k-algebras making the necessary two squares commute up to homotopy.*

An equivalent product had been given in work of Munkholm [Mun74, §9], who seemingly regarded it as a curiosity, probably ill-behaved and possibly ill-defined, but it turns out to be well-defined independent of choices and have good properties. It promotes (added hypotheses and conclusions in red) the strongest classical Eilenberg–Moore collapse result to a ring isomorphism:

Theorem 2.2 (Munkholm [Mun74], C. [Ca22a]). *Let $X \to B \leftarrow E$ be a diagram of topological spaces with $E \longrightarrow B$ a Serre fibration such that $\pi_1(B)$ acts trivially on $H^*(E;k)$ and suppose that $H^*(X;k)$, $H^*(B;k)$, and $H^*(E;k)$ are polynomial rings on at most countably many generators. If the characteristic of the principal ideal domain k is 2, assume as well that the $\smile_1$-square vanishes on some selection of polynomial generators for $H^*(X;k)$, $H^*(E;k)$, and $H^*(B;k)$. Then there is a graded k-algebra isomorphism*

$$\operatorname{Tor}_{H^*(B;k)}\big(H^*(X;k), H^*(E;k)\big) \xrightarrow{\sim} H^*(X \underset{B}{\times} E;k). \tag{2.3}$$

This result applies with $(X \to B \leftarrow E) = (* \to BG \leftarrow BK)$ to compute $H^*(G/K;k)$ with suitable coefficients, and moreover with $(X \to B \leftarrow E) = (BH \to BG \leftarrow BK)$ to compute the Borel cohomology $H^*_H(G/K;k)$, where H acts on G/K by $h \cdot gK = (hg)K$ as in Section 4.1. In particular, if H acts freely on G/K, then $H\backslash G/K$ is a smooth manifold, called a ***biquotient*** whose cohomology is the Borel H-equivariant cohomology of G/K, so the following theorem computes $H^*(H\backslash G/K;k)$:

Theorem 2.4. *Let $G > H, K$ be compact, connected Lie groups such that the orders of torsion elements of the groups $H^*(G;\mathbb{Z})$, $H^*(G;\mathbb{Z})$, and $H^*(G;\mathbb{Z})$ are invertible in the principal ideal domain k. Then there is a ring isomorphism*

$$H^*_H(G/K;k) \cong \operatorname{Tor}_{H^*(BG;k)}\big(H^*(BH;k), H^*(BK;k)\big).$$

Taking $H = 1$, one has $H^*(G/K;k) \cong \operatorname{Tor}_{H^*(BG;k)}\big(k, H^*(BK;k)\big)$.

[1] A strongly homotopy-commutative k-algebra is a technical notion that can be thought of roughly as a DGA carrying a compatible E_3-algebra structure, which is a form of homotopy-commutativity not enjoyed a mere DGA, but still low on an infinite (E_n-) hierarchy of such notions. A *map* of strongly homotopy-commutative k-algebras, on the other hand, is something *weaker* than a map of DGAS, being required only to preserve the E_3-algebra structure.

Recent work of Matthias Franz [Fra21] had already proven $H^*(G/K, k)$ had this ring structure, subject to the additional hypothesis that 2 be a unit of k, and subsequent work of Franz and myself [CaF21] had extended this to compute $H^*_H(G/K; k)$ subject to the invertibility of 2. This proof proceeded along quite different lines, using an explicitly defined product on the bar construction $\mathbf{B}(A', A, A'')$ when $A' \to A \to A''$ are maps of differential graded algebras carrying some additional operations.[2] Under weak flatness conditions, the cohomology of $\mathbf{B}(A', A, A'')$ is $\mathrm{Tor}_A(A', A'')$, so, writing $Y = X \times_B E$, we were able to leverage this product and the composite $\mathbf{B}\big(C^*(X), C^*(B), C^*(E)\big) \to \mathbf{B}\big(C^*(Y), C^*(Y), C^*(Y)\big) \to C^*(Y)$ to prove a similar collapse result. This paper closely followed the methods of Franz's paper, so that the real technical innovation was the product on $\mathbf{B}(A', A, A'')$, which it is natural to conjecture is the binary component of an A_∞-algebra structure.

Supporting this notion, in later work [Ca22b] I was able to show this product, so long as $H^*\mathbf{B}(A', A, A'') = \mathrm{Tor}_A(A', A'')$, induces Munkholm's product on Tor. One of the main results of this paper, showing the composite of two well-known natural transformations of DGAs is a third [Ca22b, Thm. 7.1], can also be used [Ca22c, §9, Fn. 14] to replace a broken lemma in the proof of a result related to Munkholm's by Husemoller–Moore–Stasheff [HuMS74, Prop. IV.5.7].

These results complete a program begun in the 1950s, and counterexamples show Theorem 2.2 is very likely to be sharp, in that suitable up-to-homotopy maps $H^*(-) \longrightarrow C^*(-)$ cannot be defined with a map of spaces $X \to B$ to make the expected square commute up to homotopy if the cohomology of the spaces is not polynomial. In recognition of this, I wrote a survey of the history and prehistory of these Eilenberg–Moore collapse results as a contribution to a conference proceedings [Ca22c].

2.1. Future work: higher homotopy structures on the free loop space

In a related direction, recent work of Manuel Rivera shows in essence that the co-Hochschild complex of the chains on a space X gives a model for the chains on its free loop space LX as a B_∞-categorical coalgebra. The homotopy Gerstenhaber algebra structure on cochains alluded to above is dual to a homotopy Gerstenhaber coalgebra structure on chains, and I would like to investigate the extent to which the former structure arises as a consequence of the latter.

3. Equivariant cohomology and K-theory

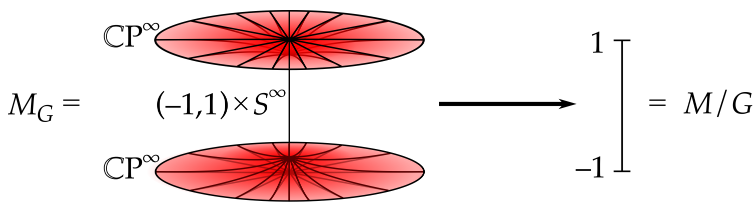
It is a well-known disappointment that the orbit space M/G of the action of a Lie group G on a topologial space M does not distinguish between orbit types; for example, when one passes to the quotient $S^2/S^1 \approx [-1, 1]$ of a standard globe S^2 under the action of the circle S^1 by rotation, both poles $*$ and latitudes S^1 become simply points. One wants to have one's cake and eat it too by taking the quotient in a way that somehow retains the distinction between orbit types, and does this via *Borel equivariant cohomology*, a central tool since its inception around 1960 [Bor+60]. One forms the ***homotopy quotient*** or *Borel construction*

$$M_G := EG \times M\big/(eg, m) \sim (e, gm),$$

where EG is the total space of the universal principal G-bundle, a contractible space with free G-action. Homotopically speaking, $EG \times M$ is no different than M, but the diagonal action on $EG \times M$ is free, so orbit types now remain distinct and we may regard M_G as a homotopically-correct replacement for M/G. The ***Borel cohomology*** $H^*_G(M)$ of the action is the singular cohomology $H^*(M_G)$ of this new construction.

[2] Called *extended homotopy Gerstenhaber algebras*, these are closely related to but distinct from strongly homotopy-commutative algebras.

For example, the homotopy quotient $(S^2)_{S^1}$ of the rotation action on the 2-sphere can be visualized as in the following cartoon.



Here forgetting the EG coordinate induces a projection to the naive quotient, whose fiber over any point of the open interval $(-1,1)$ is the (contractible) infinite-dimensional sphere $ES^1 = S^\infty$, and whose fibers over ± 1 are infinite complex projective spaces $BS^1 = \mathbb{C}P^\infty = S^\infty/S^1$. Thus M_G is homotopy equivalent to the wedge sum $\mathbb{C}P^\infty \vee \mathbb{C}P^\infty$. Its cohomology $\mathbb{Z}[x,y]/(xy)$ encodes much of the structure of the action; for example, the two fixed points show up in the fact that the ring is free of rank two over the coefficient ring $H^*_{S^1}(*) \cong \mathbb{Z}[x+y]$. In general, the orbit types can be read off of the ideal structure of $H^*_G(X)$ [Hsi75, Ch. IV], so Borel cohomology makes orbit structure legible in ring theory.

Another approach to analyzing an action studies bundles over the space. Given a G-space M, one can consider the notion of a *G-equivariant vector bundle* $V \to M$ whose total space admits a G-action such that the projection preserves the group action. These can be directly summed and tensored just as ordinary vector bundles can, and formally inverting the direct sum yields the ***equivariant K-theory*** ring $K^*_G(M)$. As in the nonequivariant case, equivariant K-theory is inherently less computable than Borel cohomology but often better-behaved algebraically.

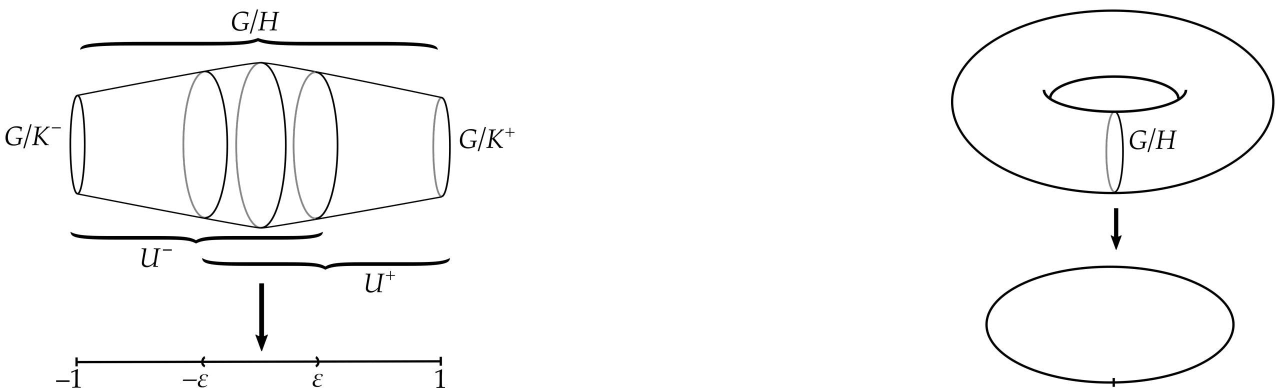
In the rest of this section we describe some of my computations.

3.1. . . . of real Grassmannians

The real Grassmannians $G_k(\mathbb{R}^n)$ of oriented k-planes in n-dimensional Euclidean space are important parametrizing objects, well-studied as manifolds in their own right. Accordingly, their rational singular cohomology rings have long been known [Ler49, Tak62][Cart51, p. 71][Bor53, p. 192]. Chen He [He16, Thms. 5.2.2, 6.3.1, Cor. 5.2.1] applied his extension of GKM-theory to odd-dimensional and nonorientable manifolds to compute the rational Borel cohomology rings of the ***isotropy actions*** on these spaces, defined as the left multiplication action of K (here $\mathrm{SO}(k) \times \mathrm{SO}(n-k)$) on the right quotient homogeneous space $G/K \approx \mathrm{SO}(n)/\big(\mathrm{SO}(k) \times \mathrm{SO}(n-k)\big)$. I showed [Ca21] that one can compute these rings much more simply using existing models and a structure result, Theorem 4.1 below.

3.2. . . . of cohomogeneity-one actions

The next simplest actions after homogeneous ones are the ***cohomogeneity-one actions***, those with one-dimensional orbit space, which are the subject of a vast geometric literature and classified in low dimensions [GGZ18]. Topologically, they all are mapping tori of G-equivariant self-homeomorphisms of homogenous spaces or double mapping cylinders of certain pairs of G-equivariant maps $G/K^- \leftarrow G/H \to G/K^+$, but they furnish many interesting examples of positively-curved manifolds with large isometry groups.



(a) M/G an interval

(b) M/G a circle

Figure 3.1: Schematic of the orbit projection $M \longrightarrow M/G$ of a cohomogeneity-one action

It is natural to explore the algebraic invariants of this class, and Oliver Goertsches, Chen He, Liviu Mare, and I computed their rational Borel cohomology [CaGHM19, Theorem 1.2]. I independently computed the equivariant K-theory of a cohomogeneity-one action later [Car22e]; the result looks similar but the proof requires substantially different techniques from Lie and representation theory. As in the cohomological case, though, the proof uses an additional structure on the Mayer–Vietoris sequence induced by the standard cover of $[-1, 1]$, which turns out to obtain in great generality.

Theorem 3.2 ([Car22e, Proposition 2.1]). *For any $\mathbb{Z}$-graded multiplicative G-equivariant cohomology theory E^*, the connecting map in the Mayer–Vietoris sequence of a triad $(X; U, V)$ of G–CW complexes with $X = U \cup V$ preserves a natural E^*X-module structure.*

This fact does not seem to appear in the literature and is needed to obtain the ring structure. Similarly [Car22e, Lemma 1.2], I prove a result computing E^* of a mapping torus with suitable coefficients, relying on an equivariant Atiyah–Hirzebruch spectral sequence; this works even when there is no transfer map because the E_2 page is Bredon cohomology [Matu73, §4], where there is always such a map.

In this paper I also discuss conditions under which a cohomogeneity-one action is K-theoretically equivariantly formal in the sense to be discussed in Section 4.2. These results have found geometric application in work of Amann–González-Álvaro–Zibrowius [AmGÁZ19, Thm. A(1)] constructing metrics of non-negative curvature on vector bundles over a class of manifolds admitting cohomogeneity-one actions.

Future work: bundles over biquotients

A related open question raised by Marcus Zibrowius is whether every vector bundle over a biquotient $K \backslash G/H$ (as discussed in Section 2), lies in the image of the natural map from the tensor product of complex representation rings $RK \otimes RH$ to $K^0_G(G/H, G/K)$, at least when the sum of the ranks of H and K equals the rank of G. From Hodgkin's Künneth spectral sequence, it follows that it will be enough to analyze $\mathrm{Tor}_{RG}(RK, RH)$.

4. Formality and equivariant formality

There is a natural map from a G-space M to its homotopy quotient M_G, given by $M \times \{e_0\} \hookrightarrow M \times EG \twoheadrightarrow (M \times EG)/\sim$ for any point $e_0 \in EG$, which is the fiber inclusion of an M-bundle $M_G \longrightarrow G\backslash EG = BG$. This fiber inclusion induces a pullback map $H^*_G(M) \longrightarrow H^*(M)$ which in the algebraic best-case scenario is *surjective*. For example, in the example of S^1 rotating S^2 above, the map

$$\mathbb{Z}[x,y]/(xy) \;\cong\; H^*_{S^1}(S^2) \longrightarrow H^*(S^2) \;\cong\; \mathbb{Z}[t]/(t^2)$$

is given by $x \mapsto t$ and $y \mapsto -t$. In this instance, the action of G on M is called ***equivariantly formal***, and a preimage $\tilde{c} \in H^*_G(M)$ of $c \in H^*(M)$ is called an ***equivariant extension*** of c. While Borel already made use of this condition in his seminar, it was given its present name by Goresky, Kottwitz, and MacPherson [GorKM98] in the paper that began what is now called ***GKM theory***. This theory allows the equivariant cohomology $H^*_T(M)$ of a ***GKM manifold***, a certain kind of well-behaved manifold with equivariantly formal action and finitely many fixed points, to be computed in terms of the combinatorics of the orbits of 0- and 1-dimensional orbits using a lemma of Chang and Skjelbred. This is the simplifying condition figuring in Theorem 5.1. Equivariant formality guarantees all classes in $H^*(X)$ admit equivariant extensions in $H^*_T(X)$, to which the Atiyah–Bott–Berline–Vergne localization theorem applies, yielding the restrictions on isotropy data mentioned in Section 5.

4.1. Equivariant cohomology and K-theory of isotropy actions

Equivariant formality simplifies the computation of equivariant cohomology. I showed the following around the time of my thesis, generalizing classical results that come to the same conclusion when $\operatorname{rk} G = \operatorname{rk} H$.

Theorem 4.1 ([Ca15, Theorem 10.1.1][Car22d, Theorem C])**.** *Let G be a compact, connected Lie group, and H a closed, connected subgroup such that the action of H on G/H is equivariantly formal.*[3] *Then there is a ring isomorphism*

$$H^*_H(G/H;\mathbb{Q}) \;\cong\; H^*(BH;\mathbb{Q}) \underset{H^*(BG;\mathbb{Q})}{\otimes} H^*(BH;\mathbb{Q}) \underset{\mathbb{Q}}{\otimes} \operatorname{im}\big(H^*(G/H;\mathbb{Q}) \to H^*(G;\mathbb{Q})\big),$$

where the $H^*(BG;\mathbb{Q})$-algebra structure on $H^*(BH;\mathbb{Q})$ is induced from the inclusion $H \longhookrightarrow G$.

Example 4.2. The group of orientation-preserving isometries stabilizing the three-plane $\mathbb{R}^3 \times \{0\}^3$ in $\mathbb{R}^6$ is $\mathrm{SO}(3) \times \mathrm{SO}(3)$. The associated $\mathrm{SO}(3)^2$-equivariant cohomology of the Grassmannian of oriented 3-planes in $\mathbb{R}^6$ is

$$\frac{\mathbb{Q}[p_1, p_1', \pi_1, \pi_1']}{(p_1 + p_1' - \pi_1 - \pi_1',\, p_1 p_1' - \pi_1 \pi_1')} \otimes \Lambda[\eta], \qquad |p_1| = |p_1'| = |\pi_1| = |\pi_1'| = 4, \quad |\eta| = 5.$$

This result implies the classical computation of $H^*(G/H;\mathbb{Q})$ in these cases. Our proof relies on a Sullivan model for biquotients due to Vitali Kapovitch [Kap04, Prop. 1][4] which also applies to homotopy biquotients [Ca21]. The model can be viewed as a compression of the Serre spectral sequence of the fibration $G \to G_{H\times H} \to BH \times BH$. Although there is no cochain-level model of equivariant K-theory, I conjectured and was eventually able to prove a related result under more stringent hypotheses [Car22d, Theorem A], which still apply up to taking a finite cover, in all cases where equivariant formality of an isotropy action is known, except those I determined in the case H is a circle. As with the cohomological case, this result generalizes the classical computations of $K^*(G/H)$ in these cases [Min75]. The K-theoretic and cohomological results are connected by a map of spectral spectral sequences from the Künneth spectral sequence in

[3] We will discuss when this hypothesis is satisfied in Section 4.3.

[4] and independently, much later, the present author

equivariant K-theory [Hod75] to that in Borel cohomology, constructed by showing one "geometric resolution" will work for both theories and applying the equivariant Chern character, which [CaF18, Thm. 5.3] identifies $H^*_G(X;\mathbb{Q})$ with the completion of $K^*_G(X;\mathbb{Q})$ with respect to IG (discussed in Section 4.2). In our case of interest, $X = Y = G/H$, and the target sequence collapses, essentially because its E_2 term is the cohomology of the Kapovitch model, which then forces the collapse of the K-theoretic sequence.

The strong commutative-algebraic hypotheses come from an unexpected source, the fact that one still does not know in general when a surjection from one finitely generated polynomial ring over $\mathbb{Z}$ to another has kernel generated by a regular sequence. Particularly, in algebro-geometric terms, we still do not know the answer to the longstanding *Abhyankar–Sathaye conjecture* addressing when a regular embedding of the affine plane $\mathbb{A}^k$ in $\mathbb{A}^n$ of affine planes can be taken by an algebraic automorphism of $\mathbb{A}^n$ to the standard embedding as $\mathbb{A}^k \times \{0\}^{n-k}$.

4.2. Weak K-theoretic equivariant formality

As it turns out [Fok19][CaF18, Thm. 5.6], equivariant formality is equivalent rationally to surjectivity of the ***forgetful map*** $f\colon K^*_G(X) \longrightarrow K^*(X)$ induced by discarding the G-structure on an equivariant bundle [MaM86]. An equivariant bundle over a point is just a representation, so $K^*_G(*)$ is the representation ring RG. The trivial G-map $X \to *$ induces a map $RG \to K^*_G(X)$, and the composition with f sends a representation to its dimension, annihilating the virtual representations IG of dimension 0. Thus f annihilates the ideal $IG \cdot K^*_G X$ of $K^*_G X$ and factors as

$$K^*_G X \longrightarrow\!\!\!\!\!\rightarrow K^*_G X \underset{RG}{\otimes} \mathbb{Z} \xrightarrow{\bar{f}} K^* X.$$

Harada–Landweber [HaL07, Prop. 4.2] observe that f is surjective if and only if $\bar{f}$ is, and say that the action is ***weakly equivariantly formal*** if $\bar{f}$ is an isomorphism. By definition, weak equivariant formality implies equivariant formality in our sense, and Fok also showed that *rationally*, weak equivariant formality is *equivalent* to equivariant formality [Fok19][CaF18, Thm. 5.6]. I was able to improve this to an integral result.

Theorem 4.3 ([Car22d, Theorem B])**.** *If a compact, connected Lie group G such that $\pi_1 G$ is free abelian acts on a compact Hausdorff space X in such a way that $K^*_G X$ is finitely generated over RG and the forgetful map $f\colon K^*_G X \longrightarrow K^* X$ is surjective, then the action is weakly equivariantly formal.*

The proof involves the map from the Atiyah–Hirzebruch spectral sequence of BG to the Atiyah–Hirzebruch–Leray–Serre spectral sequence of $X \to X_G \to BG$, which induces a tensor decomposition of the E_2 page of the former which which can be shown to persist to E_∞.

4.3. When is an isotropy action equivariantly formal?

We've now computed the Borel cohomology and K-theory of an equivariantly formal isotropy action, so it seems only fair to say *when* an isotropy action is equivariantly formal.

Question 4.4. Let G be a compact Lie group and K a closed subgroup. When is the isotropy action of K on G/K equivariantly formal?

At the beginning of 2014, only three classes of examples were known: generalized flag manifolds, those for which $H^*(G;\mathbb{Q}) \longrightarrow H^*(K;\mathbb{Q})$ is surjective, and ***generalized symmetric spaces*** [GoeN16]. In collaboration with Fok, the author was able to extend this to a complete characterization [CaF18, Thm. 1.4, Prop. 3.13] which particularly shows that if the action is equivariantly formal, then G/K is formal in the sense of rational homotopy theory. The tools involved include Kapovitch's model, a result of Shiga–Takahashi [Shi96, Thm. A, Prop. 4.1][ST95, Thm. 2.2], and classical invariant theory in the form of the Chevalley–Shepherd–Todd theorem [Kan94, p. 82].

This work follows on a string of reductions established in my dissertation [Ca19a], essentially reducing the situation to the case where G is simply-connected and K is a torus. I applied these reductions to exhaustively analyze the case when K is a circle, SO(3), or SU(2), providing an explicit algorithm. Particularly, the action is *always* equivariantly formal if K is SO(3) or SU(2). It is natural to ask if a similar classification is possible for tori S of *co*dimension one in a maximal torus of G. The result of joint work with Chen He [CaH22] is the following:

Theorem 4.5. *Let (G, S) be an pair of compact, connected Lie groups such that G is semisimple and S is a torus of codimension one in a maximal torus T of G. If $S^\perp$ is the circle orthogonal to S at $1 \in T$ under the Killing form and H is the subgroup generated by S and the commutator subgroup of the centralizer $Z_G(S^\perp)$, then (G, S) is isotropy-formal if either (a) G/H is a rational cohomology sphere or (b) G/H is a rational cohomology $S^n \times S^m$, with n even and m odd, and the number of components of the normalizer $N_G(S)$ is greater than that of $N_H(S)$.*

Characterizing when this occurs leads to a classification result for pairs (G, H) with G/H rationally homotopy equivalent to such a product of spheres and G acting irreducibly, mildly extending and revising the classification of Kramer [Kra02].

5. Equivariant complex cobordism and fixed points

One can study smooth symmetry in terms of individual manifolds or the *totality* of manifolds. ***Equivariant complex cobordism*** is one such approach; one attempts to understand when two *stably complex* G-manifolds, meaning roughly manifolds locally modeled by $\mathbb{C}^n$ or $\mathbb{C}^n \times \mathbb{R}$ and equipped with the action of a Lie group G, together bound another stably complex G-manifold, and views them as equivalent in this case. This equivalence relation makes of all stably complex G-manifolds a ring Ω^G_* which has been studied since the 1960s but is to this day only completely understood when G is an abelian p-group.

A related question attempts to characterize an action of a torus T on a stably complex manifold in terms of the normal T-equivariant bundle to the fixed-point set, (in the event of an isolated fixed point, this is just a T-representation). These ***isotropy data*** in fact determine the manifold up to equivariant cobordism and are not arbitrary, but highly interdependent by the integral localization theorem of Atiyah–Bott–Berline–Vergne (*ABBV*) [BV82, AB84], which expresses this dependency as a web of identities in the fraction field of the cohomology ring H^*BT of the classifying space. These constraints are so restrictive that one might well wonder if any family of putative normal/representation data so constrained must necessarily arise from a T-action.

Realization question (Viktor L. Ginzburg, Yael Karshon, and Susan Tolman, late 1990s)**.** *Can any abstract isotropy data satisfying all the ABBV relations be realized as the isotropy data of some torus action on a compact, oriented, equivariantly stably complex manifold?*

Elisheva Adina Gamse, Karshon, and I settled the question in the affirmative for an important class of well-behaved examples, the GKM manifolds already mentioned at the beginning of Section 4.

Theorem 5.1 ([CaGaK18])**.** *Let T be a torus. Given GKM abstract isotropy data $(X_p, \sigma_p)_{p \in P}$ satisfying the ABBV relations, there exists a compact, oriented, stably complex GKM T-manifold M with this isotropy data.*

GKM manifolds are an important setting for research, but rather special. In future work we hope to extend this result to more general actions.

In independent work analyzing the realization question in the ***semifree*** case when S^1 is a circle whose orbits are all either free or fixed points, I found the following [Ca19b].

Theorem 5.2. *Any semifree abstract isotropy data $(V_p, \sigma_p)_{p \in P}$ satisfying the ABBV identities is the isotropy data of a compact, oriented, stably complex, semifree S^1-manifold M^{2n} with isolated fixed points.*

Unexpectedly, this enables one to more constructively recover a 2004 result of Dev Sinha characterizing a case of semifree bordism.

Theorem 5.3 (Sinha [Sin05]). *Every compact, oriented, stably complex, semifree S^1-manifold with isolated fixed points is bordant to a disjoint union of direct powers of S^2 with the standard rotation action of S^1. That is, the bordism ring of such manifolds is isomorphic to the polynomial ring $\mathbb{Z}[S^2]$ on one generator.*

In the general case, even the precise statement of the realization question requires some work to nail down.

5.1. Future work: the coefficient ring

As we have mentioned, while many properties of the coefficient ring of equivariant complex cobordism are known, these rings are still not completely understood. The cohomology theories MU_A^* for A a compact, abelian Lie group have long been known to be universal equivariant complex-oriented cohomology theories [Oko82] and recent work of Hausmann [Hau22] shows that their coefficient rings are the representing (Lazard) rings for equivariant formal group laws as well, clarifying their structural role. However, in this generality, a concrete generators-and-relators–level presentation of the coefficient rings MU_G^*, which are closely related to and can be seen as a sort of a stabilization of the rings Ω_*^G, is still unkown except for certain finite groups: $MU_*^{\mathbb{Z}/2}$ (Strickland [Str01]), $MU_*^{\mathbb{Z}/p^r}$ (Hu), $MU_*^{\Sigma_3}$, Hu–Kriz–Lu [HuKL21]; Jack Carlisle has also apparently found presentations for $\Omega_*^{U:\mathbb{Z}/p}$.[5]

We hope to leverage the techniques going into these calculations and our own fixed-point techniques to give a more explicit description of MU_T^* and Ω_*^T for T a compact torus than those currently available.

5.2. Future work: claims on the embedding of complex cobordism in other rings

Parts of this work [CaGaK18] rely on an announced result (2018) of Alastair Darby that a certain abstract graph carrying GKM isotropy data arises from a stably complex manifold with a torus action by a construction examining the fixed point sets of subgroups of codimensions 0 and 1. Similarly, in 2020, Zhi Lü announced an equational characterization of the the image of the equivariant cobordism ring $\Omega_*^{U:T}$ under the map to the cobordism ring of T-equivariant disc bundles over a T-fixed base space given by taking a manifold M to the normal bundle $\nu_M(M^T)$ of its fixed point set, a result obviously also highly relevant to my program. At a certain point, if no preprints appear, it might be reasonable for me to present a proof

[5] Much more is known of the structure generally: Miščenko [Miš69] found a set of equations determining the image of $\Omega_*^{U:\mathbb{Z}/p}$ under the embedding in $\Omega_*^{U:\mathbb{Z}/p}[\mathscr{A}, \mathscr{P}]$, Kosniowski [Kos76] supplied explicit geometric generators for $\Omega_*^{U:\mathbb{Z}/p}$. Kriz [Kriz99] found an expression for $MU_*^{\mathbb{Z}/p}$ as a pullback of a *non*injective square of maps involving a localization of a quotient of a power series ring and Abram–Kriz [AbK15] found a certain algebraic expression for MU_*^A for A finite abelian.

myself so that the results of my work are supported.

6. Gelfand–Zeitlin fibers

Gelfand–Zeitlin systems are a family of completely integrable systems named for their connection to Gelfand–Zeitlin canonical bases [GS83]. They are interesting because they share many features with toric integrable systems (convexity and global action-angle coordinates) but have non-toric singularities. The fibers of their moment maps, or *Gelfand–Zeitlin fibers*, are interesting from several perspectives, such as geometric quantization [GS83, HaK14], Floer theory [NNU10, NU16, CKO20], and the topology of integrable systems on symplectic manifolds [BoMMT18, Problem 2.9]. The moment map images of Gelfand–Zeitlin systems on unitary and orthogonal coadjoint orbits are polytopes known as *Gelfand–Zeitlin polytopes* whose faces are enumerated by combinatorial diagrams called *Gelfand–Zeitlin patterns*. The unitary case is slightly easier to describe.

The initial observation is that given an $(k+1)\times(k+1)$ Hermitian matrix ξ_{k+1}, its eigenvalues $\lambda_1^{(k+1)} \geqslant \cdots \geqslant \lambda_k^{(k+1)}$ and the eigenvalues $\lambda_j^{(k)}$ of its upper-left $k \times k$ submatrix ξ_k, also in descending order, satisfy the interlacing relations

$$\begin{array}{ccccccccccccccccc}
\lambda_1^{(k+1)} & & & & \lambda_2^{(k+1)} & & & & \lambda_3^{(k+1)} & & & & \lambda_k^{(k+1)} & & & & \lambda_{k+1}^{(k+1)}. \\
& \geqslant & & \geqslant & & \geqslant & & \geqslant & & \geqslant & & \geqslant & & \geqslant & & \geqslant & \\
& & \lambda_1^{(k)} & & & & \lambda_2^{(k)} & & & & \cdots & & & & \lambda_k^{(k)} & &
\end{array} \qquad (6.1)$$

Something similar holds of the further truncations $\xi_{k-1}, \xi_{k-2}, \ldots$, leading to a triangle of inequalities. The *Gelfand–Zeitlin system* is the map on the space $\mathcal{O}$ of Hermtian matrices ξ_{k+1} with fixed eigenvalues $\lambda^{(k+1)}$ taking each matrix to the list of eigenvalues $(\lambda^{(k)}, \lambda^{(k-1)}, \ldots, \lambda^{(1)}) \in \mathbb{R}^k \times \cdots \times \mathbb{R}^1$ of all its truncations. The image of the space $\mathcal{O}$ of all such Hermitian ξ_{k+1}, which can be identified with a coadjoint orbit of the unitary group $\mathrm{U}(k+1)$, is a polytope in $\mathbb{R}^{k(k+1)/2}$ called the Gelfand–Zeitlin polytope. The fiber F over any point was known from recent work of Cho–Kim–Oh [CKO20] to be an iterated sphere bundle, and its first two homotopy groups were known. Something similar was also known due to work of Cho–Kim [CK20] in the case of a coadjoint orbit of the orthogonal group—only the matrices in question are now anti-symmetric, and eigenvalues are purely imaginary and occur in complex conjugate pairs.

Associated to the triangle of inequalities is a graph, called the ***pattern*** with one vertex for each entry and one edge for each inequality that is actually an equality, so that components represent distinct eigenvalues. This graph is known to wholly determine the diffeomorphism type of a fiber. In the unitary case, such a fiber was also known to split as a direct product of factors indexed by components of the pattern by work of Bouloc, Miranda, and Zung [BouMZ18], and Cho–Kim–Oh showed a certain free Hamiltonian torus action defined in a neighborhood of the fiber induced shown to arise from a product decomposition $F = F' \times T$ where T is the acting torus and F' is simply-connected.

In joint work with Jeremy Lane [CaL21], I refined these descriptions to show a direct product decomposition in both the unitary and orthogonal cases and to explicitly describe the diffeomorphism type of each factor. Each factor turns out to be the quotient $H\backslash G/K$ of a Lie group G by a left–right action $(h,k).g = hgk^{-1}$ of closed subgroups $H, K \leqslant G$ which can be described explicitly in terms of the pattern. This description allows us to recover all existing descriptions of individual fibers and also to compute the cohomology ring and first three homotopy groups of a fiber in both the unitary and orthogonal cases.

Theorem 6.2. *The integral cohomology of a unitary Gelfand–Zeitlin fiber is an exterior algebra on odd-degree generators. The cohomology of an orthogonal Gelfand–Zeitlin fiber over* $\mathbb{Z}[1/2]$ *is also an exterior algebra; the integral and mod-*2 *cohomology groups are isomorphic to those of a product of real Stiefel manifolds. The degrees of the*

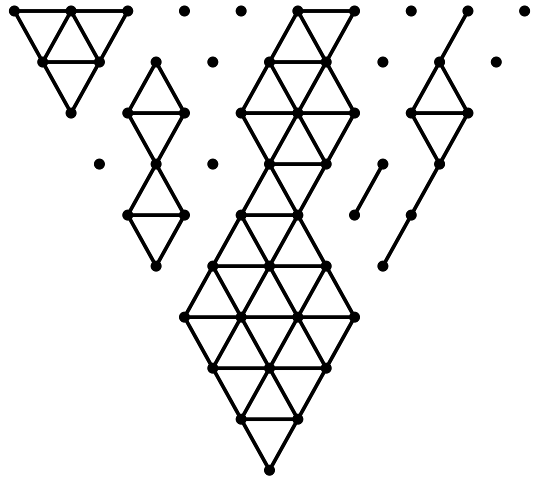


Figure 6.4: Example of a GZ pattern associated to a fiber of a GZ system on a non-regular coadjoint orbit of U(10).

relevant generators are determined in a straightforward manner by the GZ pattern.

This description also allows us to extract torus factors from a GZ fiber almost effortlessly, in both the unitary and orthogonal cases, agreeing with the previously known decomposition in the orthogonal case. Moreover, our decomposition leads to weak local expressions for a coadjoint orbit over a ray in the polytope, and this, along with the extraction of our tori, yields a topological model for a toric degeneration. That such degenerations exist is known for unitary GZ systems [NNU10], but an analogue for orthogonal GZ systems seems to remain open.

Example 6.3. Consider the GZ pattern in Figure 6.4. Using our results, one can immediately read from this pattern that an associated GZ fiber is diffeomorphic to

$$(S^1)^7 \times (S^3)^3 \times \mathrm{U}(2)\backslash\big(\mathrm{U}(4) \times \mathrm{U}(3)\big)/\mathrm{U}(2),$$

has integral cohomology ring isomorphic to

$$\Lambda[z_{1,1}, z_{1,2}, z_{1,3}, z_{1,4}, z_{1,5}, z_{1,6}, z_{1,7}, z_{3,1}, z_{3,2}, z_{3,3}, z_{5,1}, z_{5,2}, z_{7,1}], \qquad |z_{m,j}| = m,$$

and has $\pi_3 \cong \mathbb{Z}^3$.

A probable application to quantization

A future application of our work describing these fibers under development by Hamilton and Harada, is to formally prove an observation made by Guillemin and Sternberg [GS83] that over the boundary of the GZ polytope, it is precisely the GZ fibers over integral points that lie in the Bohr–Sommerfeld set.

7. Selected other work in progress

A number of other projects do not directly involve the objects discussed so far, but are generally clustered around the theme of formality.

7.1. Galois cohomology and the Bloch–Kato conjecture

The Bloch–Kato conjecture states that for a field k containing a primitive p^{th} root of unity, a certain homomorphism from the quotient $K_*^{\text{M}}(k)/(p)$ of the Milnor K-theory of k to the cohomology $H^*\big(\text{Gal}(k^{\text{sep}}/k);\mathbb{F}_p\big)$ of the absolute Galois group of k is an isomorphism. The conjecture's eventual proof due to Voevodsky relied on techniques from $\mathbb{A}^1$-homotopy theory not available at the time of its formulation and on a higher level of abstraction than one expects from the statement. A more constructive proof, or even fragments of one, might enable one to extract more of the structure of $G_k = \text{Gal}(k^{\text{sep}}/k)$ itself than is visible from the isomorphism alone.

1. For example, the isomorphism shows the cohomology groups are generated by elements of H^1, but is not explicit how to identify elements of H^n as polynomials in the elements of H^1.
2. A complete understanding of this might enable us to recover a presentation for the maximal pro-p quotient $G_k(p)$ of G_k, which has been known since work of J. Labute in the 1960s to be a *Demushkin group* in the case k is a local field, but is not well-understood even in the global case.
3. A presentation could be used to resolve a question of Positselski: the cohomology of $G_k(p)$ is known to be a quadratic algebra, but it is not known to be a Koszul algebra.
4. The *Mináč–Tân conjecture* that all n-fold Massey products of elements of $H^1\big(G_k(p);\mathbb{F}_p\big)$ vanish for $n \geqslant 3$ is known in several important cases, in particular for number fields due to recent work of Harpaz–Wittenberg, but open in general. It holds at least whenever the cochain algebra $C^*\big(G_k(p);\mathbb{F}_p\big)$ is formal. This is known not to always be the case due to counterexamples of Positselski, but all existing counterexamples arise in cases when primitive $(p^n)^{\text{th}}$-roots do not exist in k for all n, causing certain cohomology operations to be nonzero, so there remains the possibility that if this additional hypothesis were assumed, the cochain algebra would be formal and the Mináč–Tân conjecture would also hold in these cases.

Formality properties, Koszulity properties and n-Massey vanishing properties of Galois cohomology are tightly connected, but not all connections and precise implications are clear. Part of the proposed project is to clearly delineate these connections. Joint work with Ján Mináč and Federico Pasini will use techniques analogous to those effective in the computation of cohomology of homogeneous spaces and particularly formality of A_∞-algebras (as briefly discussed in Section 2) to provide a more nuts-and-bolts proof of the Bloch–Kato isomorphism and resolve several of the problems above. For context as to the developments in Galois cohomology leading to this proposal, we refer to Harpaz–Wittenberg and Mináč–Tân [HaW19, MičT2017].

7.2. The toral rank conjecture for nilmanifolds

A *nilmanifold* N is[6] a manifold which can be represented as an iterated principal torus bundle over a torus: i.e., N can be written as the total space of a principal torus bundle $T \to N \to B$ where B is again a nilmanifold. One can thus ask if it satisfies the following conjecture.

Conjecture 7.1. Let N be a space of finite topological dimension admitting an action of a torus T with finite stabilizers. Then $\dim_{\mathbb{Q}} H^*(N;\mathbb{Q}) \geqslant \dim_{\mathbb{Q}} H^*(T;\mathbb{Q})$.

Sullivan models are a common method of attack for this conjecture, which has been settled in certain special cases but remains open in general. The differentials in the Serre spectral sequence of $N \to B \to BT$ converging to $H^*(B;\mathbb{Q}) = H_T^*(N;\mathbb{Q})$ are determined by certain higher cohomology operations on

[6] among other things

$H^*(N;\mathbb{Q})$ [GorKM98, §13], and Steven Amelotte and I hope to use these operations to establish bounds on the dimension of $H^*(N;\mathbb{Q})$ and hence verify the conjecture in this case.

7.3. The Halperin conjecture for biquotients

Recall that equivariant formality is the surjectivity of the fiber restriction i^* of the bundle $M \xrightarrow{i} M_G \to BG$. One can ask the same question about other fiber bundles, and one of the principal developers and historical protagonists of rational homotopy theory, Stephen Halperin, made the following conjecture.

Conjecture 7.2. Let F be a simply-connected CW complex such that $\dim_{\mathbb{Q}}(\pi_* F \otimes \mathbb{Q})$ is finite and the Euler characteristic of F is positive. Then for any fiber bundle $F \to E \to B$, the fiber restriction $H^*(E;\mathbb{Q}) \longrightarrow H^*(F;\mathbb{Q})$ is surjective.

The conjecture was verified by Shiga and Tezuka [ST87] in the case M is a *complete flag manifold*, a homogeneous space which can be written as G/H where G and H are connected compact Lie groups and H contains a maximal torus of G. Their proof involved a careful analysis that, among other things, invoked Cartan's theorem about G/H described in Section 1. I believe that the case of a biquotient $K\backslash G/H$ with $\operatorname{rk} K + \operatorname{rk} H = \operatorname{rk} G$ will yield to a similar analysis using the Kapovitch model discussed in Section 4.1.

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